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Amalgams Which Involve Sporadic Simple Groups I

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1. INTRODUCTION

This paper and its successor [PR3] study subgroup configurations which involve groups belonging to one of the following classes:

$$\mathcal{L} = \mathcal{L}ie_1(\text{even}) \cup \mathcal{D} \cup \{S_5\}$$

and

$$\mathcal{S} = \{L \mid F^*(L)/Z(F^*(L)) \text{ is a sporadic simple group and } Z(F^*(L)) \text{ has odd order}\},$$

where

$\mathcal{L}ie_1(\text{even}) = \{L_2(2^n), U_3(2^n), SU_3(2^n), Sz(2^{2n+1}) \mid n \in \mathbb{N}\}$;

$\mathcal{D} = \{D_{2p} \mid p \text{ an odd prime}\}$ (D_{2n} is the dihedral group of order $2n$);

S_5 is the symmetric group of degree 5.

The specific situation we shall be investigating is described in

HYPOTHESIS 1.1. *Suppose that G is a group generated by proper finite subgroups P_1 and P_2 for which the following hold*:

(i) $O^{2'}(P_1)/O_2(O^{2'}(P_1)) \in \mathcal{L}$ *and* $O^{2'}(P_2)/O_2(O^{2'}(P_2)) \in \mathcal{S}$;

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(ii) $Syl_2(B) \subseteq Syl_2(P_1) \cap Syl_2(P_2)$, *where* $B := P_1 \cap P_2$;

(iii) $P_i = O^{2'}(P_i)B$ *for* $i = 1, 2$;

(iv) $C_{P_i}(O_2(P_i)) \leq O_2(P_i)$ *for* $i = 1, 2$;

(v) *B contains no nontrivial normal subgroup of G.*

For $i = 1, 2$, we put $L_i = O^{2'}(P_i)$ and $Q_i = O_2(L_i)$. Our main result which is concerned with the possible structure for P_1 and P_2 is as follows:

THEOREM A. *Assume that Hypothesis* 1.1 *holds. Then the chief factor structure of the pair* (P_1, P_2) *is known and is tabulated in Table* 1. *In particular,*

(i) $P_1/Q_1 \in \{L_2(2), S_5\}$;

(ii) $P_2/Q_2 \in \{M_{22}, \mathrm{Aut}(M_{22}), 3\,\mathrm{Aut}(M_{22}), M_{23}, M_{24}, Co_2, Co_1\}$;

(iii) $|B| \leq 2^{46}$.

In Table 1 we present information about the 12 possible pairs of subgroups (P_1, P_2) which satisfy Hypothesis 1.1. The first column presents a nomenclature for the amalgams. Column 2 gives the critical distance b associated with each amalgam (b is defined in Section 3). In column 3 we give a chief series for P_1; here, for example, $2^{1_4+2_5+1_3}L_2(2)$ is a compressed form of $2^{1+1+1+1+2+2+2+2+2+1+1+1}L_2(2)$ and (following [MS]) $\bar{4}$ represents the natural $GF(2)S_5$-module. Column 4 describes the chief series structure for P_2; here again we follow [MS] by representing the 11-dimensional (respectively, 10-dimensional) M_{23}, M_{24} (respectively, $M_{22}, \mathrm{Aut}(M_{22})$) Golay-code module by $\overline{11}$ (respectively, $\overline{10}$) and Todd-module by 11 (respectively, 10). The chief factors of dimension 12 for $3\,\mathrm{Aut}(M_{22})$, 24 for Co_1, and 22 for

TABLE 1

	b	P_1	P_2	G
$\mathcal{A}_1$	1	$2^{1_4+2_5+1_3}L_2(2)$	$2^{\overline{10}}\mathrm{Aut}(M_{22})$	Co_2
$\mathcal{A}_2$	1	$2^{1_4+2_6+1_4}L_2(2)$	$2^{\overline{11}}M_{24}$	Co_1
$\mathcal{A}_3$	1	$2^{1_5+2_4+1_3}L_2(2)$	$2^{10}M_{22}$	Fi_{22}
$\mathcal{A}_4$	1	$2^{1_5+2_4+1_4}L_2(2)$	$2^{10}\,\mathrm{Aut}(M_{22})$	$\mathrm{Aut}(Fi_{22})$
$\mathcal{A}_5$	1	$2^{1_6+2_4+1_3}L_2(2)$	$2^{11}M_{23}$	Fi_{23}
$\mathcal{A}_6$	1	$2^{1_6+2_4+1_6}L_2(2)$	$2^{11}M_{24}$	Fi'_{24}
$\mathcal{A}_7$	1	$2^{1_7+2_4+1_6}L_2(2)$	$2^{11+1}M_{24}$	Fi_{24}
$\mathcal{A}_8$	1	$2^{1_3+\bar{4}_3+1_3}S_5$	$2^{11}M_{24}$	J_4
$\mathcal{A}_8^*$	1	$2^{1_3+\bar{4}_3+1_3}S_5$	$2^{11}M_{24}$	
$\mathcal{A}_9$	2	$2^{2_1+1_5+2_5+1_3}L_2(2)$	$2_+^{1+12}3\,\mathrm{Aut}(M_{22})$	J_4
$\mathcal{A}_{10}$	2	$2^{2_1+1_{10}+2_{10}+1_8}L_2(2)$	$2_+^{1+22}Co_2$	B
$\mathcal{A}_{11}$	2	$2^{2_1+1_{11}+2_{11}+1_{10}}L_2(2)$	$2_+^{1+24}Co_1$	M

Co_2 come from the embedding of $3M_{22}$ in $SU_6(2)$, the action of Co_1 on the Leech lattice reduced mod 2, and the embedding of Co_2 in Co_1, respectively. The last column of the table gives, in all but case $\mathcal{A}_8^*$, a finite example for G which contains the given configuration.

As we see from Table 1, 8 of the 26 sporadic simple groups satisfy Hypothesis 1.1. However, the principal motivation of this work is to produce results applicable to the revision of the simple group classification. For more details of the relevance of Theorem A to such matters we refer the reader to [PR1] and [St2]. In order to enhance the usefulness of these papers for revision purposes, we have sought to make them as self-contained as possible. Our major sources for pertinent module data are [MS] and [DS] with some additional facts from [A]. While for group theoretic information about the sporadic simple groups we rely upon the Atlas [$\mathbb{A}$].

At certain points in the proof of Theorem A, the case $L_1/Q_1 \cong S_5$ gets singled out for special treatment. One such instance of this occurs in the proof of Theorem 4.2; there we need to make use of what is a special case of Theorem A which we now state as

THEOREM B. *Suppose that Hypothesis* 1.1 *holds with* $P_1/Q_1 \cong L_2(2^n)$. *Then* $P_1/Q_1 \cong L_2(2)$, $B := P_1 \cap P_2 \in Syl_2(P_1) \cap Syl_2(P_2)$, *and the pair* (P_1, P_2) *is one of the amalgams* $\mathcal{A}_1, \ldots, \mathcal{A}_7$, $\mathcal{A}_9, \ldots, \mathcal{A}_{11}$ *of Table* 1. *In particular, one of the following holds*:

(i) Q_2 *is elementary abelian and* $\operatorname{core}_{P_2}(Q_1) = 1$;

(ii) Q_2 *is extraspecial of* $+$*-type*, $\operatorname{core}_{P_2}(Q_1) = \Omega_1(Z(B)) \cong \mathbb{Z}_2$ *and, for any* $x \in P_1 \setminus B$, $\operatorname{core}_{P_2}(Q_1) \cap \operatorname{core}_{P_2^x}(Q_1) = 1$.

The proofs of Theorems A and B lead us to the basic subdivision in amalgam problems, namely, the noncommuting case and the commuting case (defined below). The main result of this paper addresses the noncommuting case.

THEOREM C. *Suppose that the noncommuting case holds for Hypothesis* 3.1. *Then* $L_\alpha/Q_\alpha \cong L_2(2)$, Q_β *is extraspecial of* $+$*-type and one of the following holds*:

(i) $L_\beta/Q_\beta \cong 3\,\mathrm{Aut}(M_{22})$, $Q_\beta \cong 2_+^{1+12}$, $\eta(L_\alpha, Q_\alpha) = 6$, *and* L_α *has a chief series described by* $2^{2+1_5+2_5+1_3}L_2(2)$.

(ii) $L_\beta/Q_\beta \cong Co_2$, $Q_\beta \cong 2_+^{1+22}$, $\eta(L_\alpha, Q_\alpha) = 11$, *and* L_α *has a chief series described by* $2^{2+1_{10}+2_{10}+1_8}L_2(2)$.

(iii) $L_\beta/Q_\beta \cong Co_1$, $Q_\beta \cong 2_+^{1+24}$, $\eta(L_\alpha, Q_\alpha) = 12$, *and* L_α *has a chief series described by* $2^{2+1_{11}+2_{11}+1_{10}}L_2(2)$.

In [PR3] we prove a companion result for the commuting case.

Since L_1/Q_1 is isomorphic to S_5 or an element of $\mathcal{D} \cup \mathcal{L}ie_1$(even), Hypothesis 1.1(ii) and (iii), together with $G \neq P_1$, imply that either $P_1 \cap P_2$ contains a unique Sylow 2-subgroup of both P_1 and P_2 or that $L_1/Q_1 \cong S_5$ and $(L_1 \cap P_2)/Q_1 \cong S_4$. We note a further point relating to this latter possibility in the case of $\mathcal{A}_8$ (of Table 1). This configuration in J_4 has $P_1/Q_1 \cong S_5$ and $P_2/Q_2 \cong M_{24}$ with $(P_1 \cap P_2)/Q_1 \cong S_4$, and using the embeddings of $P_1 \cap P_2$ into P_1 and P_2 we may construct a pair (P_1^*, P_2^*) which satisfies Hypothesis 1.1 and for which $P_1^* \cap P_2^* \in Syl_2(P_i^*)\,(i = 1, 2)$. We may think of this procedure as "pulling apart" the pair (P_1, P_2); we have denoted this pulled-apart amalgam by $\mathcal{A}_8^*$.

The remainder of this section gives an overview of the proof of Theorem A as well as introducing certain notation ([PR1] also offers a discussion of the proof of Theorem A). We observe that G, in Hypothesis 1.1, is a homomorphic image of the free amalgamated product of P_1 and P_2 over B. Since Theorem A focusses only upon the structure of P_1 and P_2 we may assume, without loss, that $G = P_1 *_B P_2$. In Section 3 the coset graph Γ of G is introduced. This yields a very useful geometric framework which enables us to analyse exhaustively the action of G upon Γ. It is well known in these so-called amalgam arguments that certain $GF(2)$-modules for the groups in $\mathcal{L}$ and $\mathcal{S}$ play a prominent, and often decisive, role. For example, the frequent occurrence of Aut(M_{22}) and $3M_{22}$ in our arguments is due to these groups possessing modules which have small centralizer indices for involutions or large quadratically acting 2-groups. Accordingly, in Section 2 we amass an extensive catalogue of module information for later use.

In Section 3, after introducing Γ, we define, for each vertex α of Γ, a normal subgroup Z_α of the vertex stabiliser G_α, the critical distance b of Γ, and the set $\mathcal{C}$ of critical pairs (of Γ). Let $(\alpha, \alpha') \in \mathcal{C}$. Then we have either $[Z_\alpha, Z_{\alpha'}] \neq 1$ (the noncommuting case) or $[Z_\alpha, Z_{\alpha'}] = 1$ (the commuting case). These two possibilities lead to different considerations. The balance of Section 3 is concerned with the proof of Theorem 3.20, which itself depends heavily on the solution of certain pushing-up problems. Among these we single out for mention Proposition 3.13 and Corollary 3.14, which are of independent interest.

With Theorem 3.20 to hand, in Section 4 we deal with the case when Γ has noncommuting critical pairs. As is always the case in this situation we find that Z_α is a failure of factorisation module for L_α/Q_α and, as the groups in $\mathcal{S}$, the Suzuki groups, and the unitary groups have no such modules, we almost immediately know that $L_\alpha/Q_\alpha \cong S_5$ or $L_2(2^n)$. We then employ Theorem 3.20 to guarantee that $Z_\tau \leq Z(L_\tau)$ for all vertices τ in Γ with $L_\tau/Q_\tau \in \mathcal{S}$. After this the bulk of the section is directed to showing that $b = 2$, $L_\alpha/Q_\alpha \cong L_2(2)$, and thus that (G, P_1, P_2, B) is an amalgam of symplectic type (see [PR2]). Most of the work here, as suggested above, is in eliminating the $L_\alpha/Q_\alpha \cong S_5$ configurations. It is at this point that we apply Theorem B, which we always assume to be proven when analysing the amalgams with $L_1/Q_1 \cong S_5$. We emphasize here that the proofs of Theorem A (for the S_5 case) and Theorem B are independent. Having shown that (G, P_1, P_2, B) is an amalgam of symplectic type the main result of Section 4, Theorem 4.2, follows from [PR2, Main Theorem]. Thus we find that the noncommuting critical pairs give rise to the amalgams $\mathcal{A}_9$, $\mathcal{A}_{10}$, and $\mathcal{A}_{11}$.

Sections 5 and 6 in [PR3] examine, respectively, the case of commuting critical pairs (α, α') when $L_\alpha/Q_\alpha \in \mathcal{L}$ and $L_\alpha/Q_\alpha \in \mathcal{S}$, respectively. In this case we know that b is odd and thus $L_{\alpha'}/Q_{\alpha'} \in \mathcal{S}$ or $\mathcal{L}$, respectively. In Section 5 after a short preliminary study we may apply two results of Stellmacher [St2], which we have recorded as Lemmas 3.9 and 3.10, to uncover the fact that $L_\alpha/Q_\alpha \cong L_2(2^n)$ or S_5. The main result of this section is Theorem 5.2, which states that there are no amalgams with commuting critical pairs and $\alpha \in O(\mathcal{L})$. At almost every turn in the analysis of this situation we are confronted by the groups $3M_{22}$, $\mathrm{Aut}(M_{22})$, and S_5. Finally we show that in a counterexample to Theorem 5.2 we must have $b = 3$, and it is the final elimination of this configuration that completes the proof of the theorem.

In Section 6 Lemma 3.10 can be applied again to show that if $L_\tau/Q_\tau \in \mathcal{L}$ and δ is a neighbor of τ in Γ, then $V_\tau := \langle Z_{\delta\tau}^G \rangle$ contains two noncentral L_τ-chief factors. This enables us once again to prove, in Lemma 6.7, that (so long as $b \geq 3$) $L_\tau/Q_\tau \cong L_2(2^n)$ or S_5 and more that $F^*(L_\alpha/Q_\alpha)/Z(F^*(L_\alpha/Q_\alpha)) \cong M_{22}$. The work in this section is then to demonstrate that the amalgams involving S_5 do not exist. This done we quickly extract Theorem 6.2.

Thus, by the end of Section 6 we have shown that if $(\alpha, \alpha') \in \mathcal{C}$ is a commuting critical pair then $b = 1$ and $L_\alpha/Q_\alpha \in \mathcal{S}$. In Section 7 we pursue this configuration relentlessly, eventually concluding (see Theorem 7.2) that (G, P_1, P_2, B) is one of the amalgams $\mathcal{A}_1$–$\mathcal{A}_8$ or $\mathcal{A}_8^*$. Two facts of importance in this section are that in a sporadic simple group H with Sylow 2-subgroup S, $\Omega_1(Z(S))$ has order 2 or $H \in \{J_1, Fi_{23}\}$ and an odd order automorphism of S has order dividing 3 or $H \cong J_1$. These two facts are proven in Lemma 2.22 and [PR4] and are used, in conjunction with bounds on the 2-rank of S, to prove in Lemma 7.10 that, if $b = 1$, then $L_{\alpha'}/Q_{\alpha'} \cong SU_3(2)$, $U_3(2)$, $Sz(2)$, $L_2(2)$, or S_5. When $L_{\alpha'}/Q_{\alpha'} \cong SU_3(2)$, $U_3(2)$, $Sz(2)$, or $L_2(2)$ we find, in Lemma 7.11, that Z_α is an $(FF + 1)$-module for L_α/Q_α and $F^*(L_\alpha/Q_\alpha)/Q_\alpha \in \{M_{22}, M_{23}, M_{24}\}$ (see below for the definition of an $(FF + 1)$-module). The next three lemmas show that, if $L_{\alpha'}/Q_{\alpha'} \not\cong S_5$, then $L_{\alpha'}/Q_{\alpha'} \cong L_2(2)$ and Proposition 7.15 reveals the amalgams $\mathcal{A}_1$–$\mathcal{A}_7$. We then move on to the $L_{\alpha'}/Q_{\alpha'} \cong S_5$ case. In Lemma 7.16 we apply Theorem B again to show that a maximal subgroup of $L_{\alpha'}$

containing a Sylow 2-subgroup normalizes Q_α; this foreshadows the "pulled apart" configuration mentioned earlier. Finally in Lemma 7.17 we unveil our last two amalgams $\mathcal{A}_8$ and $\mathcal{A}_8^*$.

In our last section, Section 8, we complete the proofs of Theorems A and B.

Our group theoretic notation is standard as given in either [Gor], [Hu], or [Suz], with the following additions. We use $E(p^n)$ to denote the elementary abelian group of order p^n. If H is a group acting on another group X, then the number of noncentral chief factors that H induces on an H-chief series in X is denoted by $\eta(H, X)$. For H a group we write $m(H)$ for the 2-rank of H, $\mathcal{A}(H) = \{A \le H \mid \Phi(A) = 1 \text{ and } m(A) = m(H)\}$, and $J(H) = \langle \mathcal{A}(H) \rangle$ is the elementary abelian version of the Thompson subgroup. If V is a nontrivial $GF(2)H$-module and A is a subgroup of H with $[V, A, A] = 0$ and $[V, A] \neq 0$, then we say A acts quadratically on V. If $|AC_H(V)/C_H(V)| \ge 4$, then A is called a quadratic subgroup and V is called a quadratic module for H. Assume that A acts quadratically on the $GF(2)H$-module V; if, furthermore, $1 \neq [V : C_V(A)] \le |AC_H(V)/C_H(V)|$, then V is called an FF-module for H (failure of factorisation module). While if $|AC_H(V)/C_H(V)|$ is elementary abelian, though not necessarily quadratic, and $1 \neq [V : C_V(A)] \le 2|AC_H(V)/C_H(V)|$, then V is called an $(FF + 1)$-module for H. If V is an FF-module (respectively, $(FF + 1)$) and A satisfies $1 \neq [V : C_V(A)] \le |AC_H(V)/C_H(V)|$ (respectively, $1 \neq [V : C_V(A)] \le 2|AC_H(V)/C_H(V)|$), then A is called an offending (respectively, an $(FF + 1)$ offending) subgroup. For a particular sporadic simple group we use the Atlas names for conjugacy classes.

2. $GF(2)$-REPRESENTATIONS

This section gathers together and proves results about the structure and $GF(2)$-representation of the elements of $\mathcal{L} \cup \mathcal{S}$. Our first theorem is from [MS], given in a shortened form. We recommend that the reader, while reading this section, have a copy of this article as well as a copy of the atlas [$\mathbb{A}$] to hand.

THEOREM 2.1 [MS, Theorems 1 and 3]. *Suppose that $H \in \mathcal{S}$ is a faithful irreducible quadratic module for H. Set $G = \langle F | F$ is a quadratic subgroup of $H \rangle$. Then*

(i) $G \cong M_{12}$ *and* $\dim_{GF(2)} V = 10$;

(ii) $G \cong \mathrm{Aut}(M_{12})$ *and* $\dim_{GF(2)} V = 10$;

(iii) $G \cong \mathrm{Aut}(M_{22})$ *and* $\dim_{GF(2)} V = 10$;

(iv) $G \cong 3M_{22}$ *and* $\dim_{GF(2)} V = 12$;
(v) $G \cong M_{24}$ *and* $\dim_{GF(2)} V = 11$;
(vi) $G \cong J_2$ *and* $\dim_{GF(2)} V = 12$;
(vii) $G \cong Co_1$ *and* $\dim_{GF(2)} V = 24$;
(viii) $G \cong Co_2$ *and* $\dim_{GF(2)} V = 22$;
(ix) $G \cong 3Suz$ *and* $\dim_{GF(2)} V = 24$.

The structure of the above mentioned quadratic modules will be revealed more fully later in this section.

LEMMA 2.2. *Suppose that* $G \in \mathbf{S}$. *Then an upper bound for the* 2-*rank*, $m = m(\mathrm{Aut}(G'/Z(G')))$ *is as indicated in the following table*:

$G'/Z(G')$	m	$G'/Z(G')$	m	$G'/Z(G')$	m
M_{11}	2	McL	4	Co_1	11
M_{12}	4	Ly	4	Fi_{22}	10
M_{22}	5	HS	5	Fi_{23}	11
M_{23}	4	He	6	Fi'_{24}	12
M_{24}	6	Suz	6	HN	6
J_1	3	Ru	6	Th	5
J_2	4	$O'N$	4	B	22
J_3	4	Co_3	4	M	24
J_4	11	Co_2	10		

Proof. See [A, Table 11.2].

Later in this section we shall need to draw on properties of the following "small" $GF(2)$-modules which we now describe. Recall that A_6, respectively, S_6 contains two conjugacy classes of subgroups isomorphic to A_5, respectively, S_5. Thus A_6 and S_6 possess two nonisomorphic permutation modules of dimension 6, each of which contains a composition factor of dimension 4. In either case we call the four-dimensional module arising in this way a natural A_6, respectively, S_6 module; see [LPR] for further details. Properties of the six-dimensional irreducible modules for $3A_6$ and $3S_6$ may be gleaned from [R]. For A_7 and A_8, the permutation module is the six-dimensional composition factor of the seven- (respectively, 8-) dimensional permutation module. By the four-dimensional natural A_8-module we mean either the module or its dual obtained via the isomorphism $A_8 \cong GL_4(2)$. The relevant calculations for these A_7- and A_8-modules are omitted and left to the reader.

We now turn to examine the irreducible $GF(2)S_5$-modules. Recall that S_5 has three nonisomorphic modules over $GF(2)$ (by [HB, 3.11]): the trivial module, the natural module, and the orthogonal module. Both the natural module and the orthogonal module are four-dimensional over $GF(2)$. The natural module arises from the natural two-dimensional $GF(4)$-module for $L_2(4) \cong A_5$ viewed as a $GF(2)$-module with the transpositions from G induced field automorphisms of $GF(4)$. The orthogonal module is the "natural" module obtained from the isomorphism $S_5 \cong GO_4^-(2)$; however, for calculations it is usually best to consider it as the space spanned by the vectors of even weight in the five-dimensional permutation module. We observe that the orthogonal module remains irreducible on restriction to $A_5 \cong L_2(4)$, this restricted module is not isomorphic to a natural $L_2(4)$-module, rather it is projective and can be identified with the $L_2(4)$ Steinberg module, and we will call it the orthogonal $L_2(4)$-module. The next result is well known and may be verified by straightforward calculations.

LEMMA 2.3. *Suppose that $G \cong S_5$ and set $H = G'$. Assume that V is a nontrivial irreducible $GF(2)G$-module, $S \in Syl_2(G)$, $F_1 = S \cap H$, F_2 is the fours subgroup of S not in H, $C_4 \cong C \leq S$, and t is an involution in S. Then*

(i) $|C_V(S)| = 2$;

(ii) *if $t \in H$, then $[V : C_V(t)] = 4$;*

(iii) *if $[V : C_V(t)] = 2$, then V is an orthogonal module and $t \notin H$;*

(iv) *$C_V(S) \leq [V, t]$ and $C_V(t) \neq [V, S]$ for all involutions t in G;*

(v) *if V is an orthogonal module, then*

(a) $[V, F_2] = C_V(F_2) = [V, S; 2] \cong E(2^2)$;

(b) *$[V, S] = [V, F_1] = [V, C] \cong E(2^3)$ and $C_V(S) = C_V(F_1) = [V, F_1; 2] = [V, S; 3] \cong E(2)$;*

(c) $C_G(C_V(S)) = N_G(F_1) \cong S_4$;

(d) $C_G([V, F_2]) = F_2$;

(e) *if $g \in G$ and $[V, F_2] \cap [V, F_2^g] \neq 0$, then either $[V, F_2] = [V, F_2^g]$ and $F_2^g = F_2$, or $\langle F_2, F_2^g \rangle \cong S_4$ and $C_V(F_2) \cap C_V(F_2^g) = C_V(\langle F_2, F_2^g \rangle)$;*

(f) *the elements of order 3 in G have two-dimensional fixed point space on V;*

(vi) *if V is a natural module, then*

(a) $C_V(F_1) = [V, F_1] = [V, S; 2] \cong E(2^2)$;

(b) *$[V, S] = [V, F_2] = [V, C] \cong E(2^3)$ and $C_V(S) = C_V(F_2) = [V, F_2; 2] = [V, S; 3] \cong E(2)$;*

(c) $C_G(C_V(s)) = N_G(F_2) = S$;

(d) $C_G([V, F_1]) = F_1$;

(e) *if $g \in G$, then $[V, F_1] = [V, F_1^g]$ or $[V, F_1] \cap [V, F_1^g] = 0$;*

(f) *G operates transitively on the nonzero vectors of V and the elements of order 3 in G act fixed point freely on $V^{\#}$;*

(g) *for all $1 \neq X \leq S$, $[V, X] \geq C_V(S)$.*

LEMMA 2.4. *Suppose that* $G \cong S_5$, $E(2^2) \cong F \leq S \in Syl_2(G)$, V *is a* $GF(2)S_5$*-module, with* $\eta(G,V) = 1$ *and* $[V,F,F] = 0$. *If* $V = \langle Z^G \rangle$, *where* $Z \leq C_V(F)$ *is S-invariant, then either* $|Z| \leq 2^2$ *or* $Z \cap C_V(G) \neq 0$.

Proof. Supposing that $|Z| \geq 2^3$ and $Z \cap C_V(G) = 0$ we derive a contradiction. Then without loss of generality we may assume that $C_V(G) = 0$. Let V_0 be a minimal G-invariant subspace of V. Since $\eta(G,V) = 1$ and $V = \langle Z^G \rangle$, $V = ZV_0$ and $|V_0| = 2^4$. Choose $g \in G$, such that $\langle F, F^g \rangle \geq O^2(G)$. From Lemma 2.3(v) and (vi), $[V_0 : C_{V_0}(F)] = 2^2$ and hence $[V : C_V(F)] = 2^2$ with $|C_V(F)| \geq 2^3$. But this then forces $C_V(O^2(G)) \neq 0$, whence $C_V(G) \neq 0$, a contradiction.

Our next lemma is useful for providing quadratically acting subgroups.

LEMMA 2.5. *Suppose that* V *is a faithful* $GF(2)G$*-module and suppose that* $1 \neq E$ *is a quadratically acting subgroup of* G.

(i) *If* $t \in C_{C_G(E)}(V/C_V(E)) \setminus E$ *is an involution, then* $\langle E, t \rangle$ *is a quadratic subgroup of* G.

(ii) *If* F *is a maximal quadratically acting subgroup of* G *and* $T \geq \Omega_1(C_T(F)) > F$, *then* $[V,F,T] \neq 0$ *and* $[V,T,F] \neq 0$.

Proof. (i) By assumption $[V,t] \leq C_V(E)$, and so $[V,t,E] = 0$. Since we also have $[t,E,V] = 0$, the three subgroup lemma gives $[V,E,t] = 0$ and thus

$$[V,t][V,E] \leq C_V(t) \cap C_V(E).$$

Hence (noting that $|\langle E,t \rangle| \geq 4$) (i) follows.

Now suppose F is maximal and choose $t \in C_T(F) \setminus F$ an involution. If $[V,F,T] = 0$, then $[V,F,t] = 0$ whence, using $[F,t,V] = 0$ and the three subgroup lemma, $[t,V,F] = 0$. So $t \in C_{C_G(F)}(V/C_V(F)) \setminus F$ and now part (i) contradicts the maximality of F. Thus $[V,F,T] \neq 0$ and a similar argument gives $[V,T,F] \neq 0$.

LEMMA 2.6. *Suppose that* G *is a finite group which contains a subgroup* $H \cong A_6$ *in which the distinct conjugacy classes of elements of order* 3 *in* H *are fused in* G. *If* $t \in H$ *is an involution and* V *is a* $GF(2)G$*-module with* $C_G(V) \leq O(G)$, *then* $[V : C_V(t)] \geq 2^4$.

Proof. Suppose that G, H, t, and V are as in the statement of the lemma. Furthermore, assume that $\eta(H,V) = 1$ and that this noncentral H-chief factor is a natural $GF(2)H$-module. Then the different classes of elements of order 3 in H have different commutator sizes on the module V. Thus, since the elements of order 3 in H are fused in G, we have a contradiction. Hence either $\eta(H,V) = 1$ and the noncentral chief factor is not a natural $GF(2)H$-module or $\eta(H,V) \geq 2$. Using the fact that the centralizer in a four-dimensional (respectively, 16-dimensional) $GF(2)A_6$-module of an involution in A_6 is two-dimensional (respectively, eight-dimensional), we deduce that Lemma 2.6 holds.

PROPOSITION 2.7. *Suppose that $G \in \mathcal{S}$ is not perfect and that V is a $GF(2)G$-module with $C_G(V) \leq O(G)$. If $t \in G \backslash G'$ is an involution, then $[V: C_V(t)] \geq 2^4$ or V is a quadratic module for G and t is contained in a quadratic fours group on V.*

Proof. Suppose that G, t, and V are as in the statement of the proposition and suppose that $[V : C_V(t)] \leq 2^3$; we will show that V is a quadratic module for G. Since t is an involution in $G \backslash G'$ and, by [$\mathbb{A}$], $[G : G'] = 2$, $C_G(t) = C_{G'}(t) \times \langle t \rangle$. Now, by [$\mathbb{A}$], either $|C_{G'}(t)|_2 \geq 2^4$ or $G/O(G) \cong \mathrm{Aut}(M_{12})$ and $C_G(t) \cong S_5 \times 2$, or $G/O(G) \cong \mathrm{Aut}(O'N)$ and $C_G(t) \cong J_1 \times 2$. Thus, as $C_G(t)/C_{C_G(t)}(V/C_V(t))$ must embed into $GL_3(2)$, we have in all cases that $|C_{C_{G'}(t)}(V/C_V(t))|_2 \neq 1$; therefore, Lemma 2.5(i) gives the result.

PROPOSITION 2.8. *Suppose that $G \in \mathcal{S}$ and that V is a $GF(2)G$-module with $C_G(V) \leq O(G)$. If $t \in G'$ is an involution, then either $[V : C_V(t)] \geq 2^4$, or V is a quadratic module for G' and t is contained in a quadratic fours group on V.*

Proof. We suppose that $[V : C_V(t)] \leq 2^3$ and put $L = F^*(G)/O(G)$ and $K = C_{C_G(t)}(V/C_V(t))$. Then $C_G(t)/K$ embeds in $GL_3(2)$. We seek to show that V must be a quadratic module for G'. Observe that this will follow from Lemma 2.5(i) whenever we have $m(C_L(t)) \geq 4$. If L contains a subgroup H such that $[L : H]$ is odd and H contains a normal elementary abelian 2-subgroup of rank greater than or equal to 5, then it follows that $m(C_L(t)) \geq 4$. Hence we deduce that $m(C_L(t)) \geq 4$ when L is isomorphic to one of $M_{24}, He, Co_2, Fi_{22}, Th, Fi_{23}, Co_1, J_4, Fi'_{24}, B, M$ (where H is, respectively, $2^6 : 3S_6$, $2^6 : 3S_6$, $2^{10} : \mathrm{Aut}(M_{22})$, $2^{10} : M_{22}$, $2^{5\cdot}L_5(2)$, $2^{11\cdot}M_{23}$, $2^{11} : M_{24}$, $2^{10} : L_5(2)$, $2^{11\cdot}M_{24}$, $2^9.2^{16}.S_8(2)$, $2^{10+16\cdot}O^+_{10}(2)$). If L is isomorphic to one of $J_2, HS, Ru, Suz, Co_3, HN$, then L has two conjugacy classes of involutions one of which is not a 2-central class. Since $m(C_L(s)) \geq 4$ for a non-2-central involution s (by virtue, respectively, of the following subgroups: $2^2 \times A_5, 2 \times A_6 2^2, 2^2 \times Sz(8), 2^2 \times L_3(4), 2 \times M_{12}, 2^{6\cdot}U_4(2)$), again we get $m(C_L(t)) \geq 4$. If L is isomorphic to one of $M_{22}, M_{23}, J_3, McL, Ly$, then L has just one conjugacy class of involutions and, therefore, $m(C_L(t)) \geq 4$ (because, respectively, of the following subgroups: $2^4 : A_6, 2^4 : A_7, 2^4 : (3 \times A_5), 2^4 : A_7, 3^{\cdot}\mathrm{Aut}(McL)$).

It remains to examine the possibilities M_{11}, M_{12}, J_1, and $O'N$. For L isomorphic to M_{12} (with $t \in 2A$), $J_1, O'N$ we have, respectively, $C_L(t) \cong$

$2 \times S_5, 2 \times A_5, 4^{\cdot}L_3(4):2$. Since $C_G(t)/K$ embeds in $GL_3(2)$, Lemma 2.5 applies to give that V is a quadratic module. Finally when $L \cong M_{11}$ or M_{12} (with $t \in 2B$) we may appeal to Lemma 2.6 and obtain $[V : C_V(t)] \geq 2^4$, contrary to our assumption $[V : C_V(t)] \leq 2^3$.

This completes the verification of Proposition 2.8.

PROPOSITION 2.9. *Suppose that $G \cong M_{12}$ or* Aut(M_{12}) *and that V is a nontrivial irreducible quadratic module for G with F a maximal quadratic subgroup of G. Then $|F| = 4$, $[V : C_V(t)] \geq 2^4$ for all involutions t in F and $[V : C_V(F)] \geq 2^5$.*

Proof. From [MS, Theorem 2(a), (b)], there are two possibilities for F. In both cases $|F| = 4$. We consider first the case when $F \leq G'$. Then, by [MS] and [$\mathbb{A}$], the involutions are all of class $2B$. Thus Lemma 2.6 applies and we have, for all involutions of $t \in F$, $[V : C_V(t)] \geq 2^4$. Now if $C_V(t) = C_V(F)$, then $C_V(F)$ is invariant under $\langle N_G(F), C_G(t)\rangle \geq G'$, which is a contradiction. Thus the proposition holds in this case. Now suppose that $F \nleq G'$. Then, by [MS, Theorem 2] and [$\mathbb{A}$], the involutions in $F \setminus G'$ are both of class $2C$ and invert an element of order 11 in an $L_2(11)$ subgroup of M_{12}. Thus (as $\dim_{GF(2)} V = 10$) for $t \in F \setminus G'$, $[V : C_V(t)] \geq 2^5$ and so $[V : C_V(F)] \geq 2^5$. So it remains to consider the case when $t \in F \cap G'$. Then $t \in 2A$ and $C_{G'}(t) \cong 2 \times S_5$. If $[V : C_V(t)] \leq 2^3$, then Proposition 2.8 yields that G' contains a quadratic fours group which contains a $2A$ involution, contrary to [MS, Theorem 2(a)]. Therefore, $[V : C_V(t)] \geq 2^4$, and we have the proposition.

PROPOSITION 2.10. *Suppose that $G \cong M_{24}$ and that V is a nontrivial irreducible quadratic module for G with F a maximal quadratic subgroup of G. Then*

(i) $|F| = 4$;

(ii) *if t is an involution of class $2A$ in G, then $[V : C_V(t)] = 2^4$;*

(iii) *if t is an involution of class $2B$ in G, then $[V : C_V(t)] = 2^5$;*

(iv) $[V : C_V(F)] \geq 2^5$.

Proof. That (i) holds is straight from [MS, Theorem 2]. Since $V \cong 11$ or $\overline{11}$, it suffices to prove (ii) and (iii) for $V \cong 11$ (the Todd module). Now suppose that t is an involution in G of class $2A$. Then $C_G(t) \cong 2^{1+(3+\bar{3})} : L_3(2)$ is a maximal subgroup of $H \cong 2^4 : A_8$ and we may assume without loss of generality that $t \in O_2(H)$. From [MS, Lemma 3.5], $\dim_{GF(2)} C_V(O_2(H)) = 6$ and the chief factors of $C_G(t)$ in V have dimensions 3, 3, 3, 1, and 1. Also $V/C_V(t)$ contains at least one central chief factor. Using $V/C_V(t) \cong [V, t]$ (as $C_G(t)$-modules) and $[V, t] \leq C_V(t)$ we conclude that either $[V : C_V(t)] = 2$ or $[V : C_V(t)] = 2^4$. The former possibility, as $O_2(H)$ is generated involutions of class $2A$, yields $[V:C_V(O_2(H))] \le 2^4$ against $\dim_{GF(2)} C_V(O_2(H)) = 6$. Thus $[V:C_V(t)] = 2^4$, proving part (ii).

Now suppose that t is of class $2B$ and suppose that $[V:C_V(t)] \le 2^4$. Then, because $C_G(t) \cong 2^{(1+1+4)}:S_5 \le 2^6:3S_6$ (here $O_2(C_G(t)) \cong 2^{(1+1+4)}$ is a uniserial S_5-module and $C_G(t)' = O^2(C_G(t)) \ge O_2(C_G(t)))$ either $[[V,t], O^2(C_G(t))] = 0$, or $\|[V,t]\| = 2^4$ and, as a module, is irreducible for $C_G(t)/O_2(C_G(t)) \cong S_5$. Thus in either case we observe that $[[V,t], O_2(C_G(t))] = 0$. Hence, by the three subgroup lemma, $[[V, O_2(C_G(t))], t] = 0$. So closing under $N_G(O_2(C_G(t))) \cong 2^6:3S_6$, we get $[[V, O_2(C_G(t))], O_2(C_G(t))] = 0$, which is contrary to part (i). Thus we have $[V:C_V(t)] \ge 2^5$ and hence equality, as $\dim_{GF(2)} V = 11$. This completes (iii).

Now suppose that (iv) is false. Then $[V:C_V(F)] \le 2^4$ and, using parts (ii) and (iii), if $t \in F^\#$, then $C_V(t) = C_V(F)$ has codimension 4 and t is of class $2A$. But then, by [MS, Theorem 2], $N_G(F) = 2^{2.1+2.2+1.2}(S_3 \times S_3)$, so $C_V(F)$ is invariant under $\langle C_G(t), N_G(F)\rangle = G$, which is absurd.

PROPOSITION 2.11. *Suppose that* $G \cong \mathrm{Aut}(M_{22})$ *and that* V *is a nontrivial irreducible quadratic module for* G *with* F *a maximal quadratic subgroup of* G. *Then*

(i) $|F| = 4$;

(ii) $F \nleq G'$ *and the elements of* $F \setminus G'$ *are of class* $2B$;

(iii) *for* $t \in F \cap G'$, $[V:C_V(t)] = 2^4$;

(iv) *for* $t \in F \setminus G'$, $[V:C_V(t)] = 2^3$;

(v) $C_V(F) = [V,F]$ *and* $\|[V,F]\| = [V^* : C_{V^*}(F)] = 2^4$ *or* 2^5 (*both occur*!);

(vi) $C_G(C_V(F)) = C_G([V,F]) = F$;

(vii) *if* $S \in Syl_2(G)$ *and* $z \in Z(S)^\#$, *then* $[V,S,S,z] \ne 0$.

Proof. Using [MS, Theorem 2] and [$\mathbb{A}$] we have (i) and (ii). Now let $t \in F \cap G'$. Then Lemma 2.6 applies and so $[V:C_V(t)] \ge 2^4$. If $[V:C_V(t)] = 2^5$, then $C_V(t) = [V,t]$ and $\eta(C_G(t), V) = 2 \cdot \eta(C_G(t), [V,t]) = 2$ or 4; however, [MS, Theorems 2(c) and 3] indicate that $\eta(C_G(t), V) = 3$, which is a contradiction. Therefore, (iii) holds. Next, if $t \in F \setminus G'$, then $P = C_G(t) \cong 2 \times (2^3 L_3(2))$ and [MS, Lemma 3.3(b)] describes the restriction of V to P. Because $V/C_V(t) \cong [V,t]$ as a $GF(2)P$-module, we deduce that (iv) holds.

Now we embark on a proof of (v); by (iii), we have, for $x \in F \cap G'$,

$$E(2^6) \cong C_V(x) \ge C_V(F) \ge [V,F] \ge [V,x] \cong E(2^4).$$

If $C_V(F) = [V, F] \cong E(2^5)$, then as, by [MS, Theorem 2(c)], F is normalized by $C_G(x)$ and $\eta(C_G(x), V)$ is even, we reach the same contradiction as in part (iii). Hence, if (v) is false then we must have $C_V(x) = C_V(F) \cong E(2^6)$ and $[V, x] = [V, F] \cong E(2^4)$. Let t_1 and t_2 be the distinct elements of $F \setminus G'$, $C_1 = C_G(t_1)$ and $C_2 = C_G(t_2)$. Then $C_1 \cap C_2$ is a maximal subgroup of both C_1 and C_2 of shape $2^{1+1+2}S_4$. Now, $[V, F] = [V, t_1][V, t_2]$ and $C_V(F) = C_V(t_1) \cap C_V(t_2)$, so, because $\|[V, F]\| = 2^4 = [V : C_V(F)]$, $\|[V, t_1] \cap [V, t_2]\| = [V : C_V(t_1)C_V(t_2)] = 2^2$. However, $C_1 \cap C_2$ normalizes both $[V, t_1] \cap [V, t_2]$ and $C_V(t_1)C_V(t_2)$, this then contradicts the fact that $V/C_V(t_1) \cong [V, t_1]$ as a $C_G(t_1)$-module; therefore, (v) holds.

Set $E = C_G([V, F])$, and let $t \in F \setminus G'$. Then clearly $[V, t]$ is centralized by E. From [$\mathbb{A}$], $C_G(t)$ is a maximal subgroup of G and so is the stabilizer in G of $[V, t]$. Thus $E \leq C_G(t)$. Further, $\eta(C_G(t), [V, t]) = 1$ then forces $E \leq O_2(C_G(t)) \cong E(2^4)$. Therefore E is elementary abelian and $[E, F] = 1$ whence part (vi) now follows from Lemma 2.5 and the maximality of F.

Finally, assume that $S \in Syl_2(G)$, $z \in Z(S)^{\#}$, and $[V, S, S, z] = 0$. Then the three subgroup lemma implies that $[V, S, z, S] = 0$. Using [MS, Lemma 3.3(b)] (and dualizing) we observe that $|V/[V, S]| = 2 = |C_V(S)|$, which implies $[V : C_V(z)] \leq 2^2$, contrary to (iii) and (iv). Thus (vii) holds.

PROPOSITION 2.12. *Suppose that $G \cong 3M_{22}$ and that V is a faithful nontrivial irreducible quadratic module for G. Let $S \in Syl_2(G)$, let F be a maximal quadratic subgroup of G, and let F_1 be a subgroup of F of order at least* 4. *Then the following hold*:

(i) $|F| = 2^3$ *and* $N_G(F) \cong 2^3 : L_3(2) \times 3$, *or* $|F| = 2^4$ *and* $N_G(F) \cong 2^4 3A_6$.

(ii) *for* $t \in F^{\#}$, $[V : C_V(t)] = 2^4$.

(iii) $[V : C_V(F_1)] = 2^6$, $C_V(F_1) = [V, F_1]$, *and* $C_G(C_V(F_1)) = F$, *where F is the maximal quadratic group containing F_1.*

(iv) $[V, S, S, F_1] \neq 0$.

(v) *Assume that $F_1 \cong E(2^3)$ and that W is an F_1-invariant subspace of V of codimension* 4. *Then* $\|[W, F_1]\| \geq 2^2$.

(vi) *Suppose that* $W \geq C_V(F)$, $|F| = 2^4$, *and* $[V : W] = 2^2$. *Then* $\|[W, F]\| \geq 2^3$.

(vii) *If $F \leq S_1 \in Syl_2(G)$ and $|F| = 2^4$, then $F \lhd S_1$.*

(viii) *Assume that $H \cong 3\,\mathrm{Aut}(M_{22})$ and U is a faithful $GF(2)H$-module which when restricted to $F^*(H)$ is an irreducible quadratic $GF(2)F^*(H)$-module. If t is an involution in $H \setminus F^*(H)$, then $[U : C_U(t)] = 2^6$.*

Proof. Part (i) is straight from [MS, Theorem 2(d)]. Suppose that $t \in F^{\#}$. Then, as G contains only one conjugacy class of involutions,

$[V : C_V(t)] \geq 2^4$ follows from the structure of V restricted to $2^4 3A_6$ given in [MS, Theorem 3] (using $[W : C_W(s)] = 2^2$ for an involution, s, in $3A_6$ acting on its six-dimensional irreducible module W). Now, if $[V : C_V(t)] > 2^4$, then, since G preserves a $GF(4)$-structure via the action of $Z(G)$, we get $[V : C_V(t)] = 2^6$. However, using [MS, Lemma 3.4 and notation] and choosing $t \in Q_1 \cap Q$, we find that $C_V(t)$ is invariant under $\langle 2^4 3A_6, 2^3 : L_3(2) \times 3 \rangle = G$, which is a contradiction. Hence (ii) is true.

Now suppose that $t_1, t_2 \in F_1^{\#}$ with $t_1 \neq t_2$. If $[V : C_V(F_1)] \leq 2^5$, then, as before, $[V : C_V(F_1)] = 2^4$ and $C_V(F_1) = C_V(t_1) = C_V(t_2)$ by (ii). But then, from [$\mathbb{A}$], $C_V(F_1)$ is invariant under

$$\langle C_G(t_1), C_G(t_2) \rangle \in \{3M_{22}, 2^4 : 3A_6, 2^{\bar{4}}S_5 \times 3\}.$$

All these cases are impossible by [MS, Lemma 3.4] and so $[V : C_V(F_1)] \geq 2^6$. Note that

$$[V, t_1][V, t_2] \leq [V, F_1] \leq [V, F] \leq C_V(F) \leq C_V(F_1).$$

From part (ii) and the action of $Z(G)$ on V, either $[V, t_1] = [V, t_2] \cong E(2^4)$ or $[V, t_1][V, t_2] = C_V(F_1)$. The former possibility, just as before, is contradicted by [MS, Lemma 3.4]. Therefore, $C_V(F_1) = C_V(F) = [V, F] = [V, F_1]$. By [$\mathbb{A}$], $2^3 : L_3(2) \times 3$ and $2^4 3A_6$ are maximal subgroups of G and so $N_G(F)$ is the stabilizer in G of $C_V(F) = C_V(F_1)$. Hence $C_G(C_V(F_1)) \leq N_G(F)$ and then $C_G(C_V(F_1)) \leq O_2(N_G(F)) = F$, so proving part (iii).

Next suppose that $[V, S, S, F_1] = 0$. Then, because $C_V(F_1) = C_V(F)$, $[V, S, S, F] = 0$, whence, as the three subgroup lemma implies that $[V, [S, S]] \leq [V, S, S]$, $[V, [S, S], F] = 0$. Therefore $[S, S] \leq C_G(V/C_V(F)) = F$. This, however, implies that the Sylow 2-subgroups of $N_G(F)/F$ are abelian, which is contrary to (i). Hence (iv) holds.

Now we prove (v). Suppose that $|F_1| = 2^3$, W is an F_1-invariant subspace of V of codimension 4, and that $\|[W, F_1]\| \leq 2$. Then, by (iii), $[W, F_1]$ has order 2; thus $[W, F_1]$ is not a $GF(4)$-subspace of V. It follows that W is also not a $GF(4)$-subspace; in particular, $W \neq C_V(t)$ for any $t \in F^{\#}$. Since $[W, F_1]$ has order 2 we deduce that $[W : C_W(t)] = 2$ for any $t \in F_1^{\#}$. Let t_1, $t_2 \in F_1^{\#}$ with $t_1 \neq t_2$. If $C_W(t_1) = C_W(t_2)$, then $[V : C_V(\langle t_1, t_2 \rangle)] \leq 2^5$, which is against (iii). Thus $[V : C_W(t_1) \cap C_W(t_2)] = 2^6$ and so, by (iii),

$$C_W(t_1) \cap C_W(t_2) = C_V(\langle t_1, t_2 \rangle) = C_V(F_2) = C_V(F_1),$$

for any fours group $F_2 = \langle t_1, t_2 \rangle$ in F_1. Now since $[W : C_W(F_1)] = 2^2$ and $|F_1| = 2^3$, we conclude that there exists $s_1, s_2 \in F_1^{\#}$ with $s_1 \neq s_2$ such that $C_W(s_1) = C_W(s_2)$, which is a contradiction. This concludes the proof of (v).

Now assume that $|F| = 2^4$ and W is a subspace of V of codimension 2 containing $C_V(F)$. Note that W is F-invariant. Suppose that, contrary to (vi), $\|[W, F]\| \le 2^2$. Then for each $x \in F^{\#}$, by (ii), $[W, x] = [W, F]$ and consequently $C_V(x) \le W$. Therefore, using (iii), $W = \prod_{x \in F^{\#}} C_V(x)$. So W is preserved by the action of $Z(G)$ and hence is a $GF(4)$-subspace. Now we consider the two-dimensional $GF(4)$-space $W/C_V(F)$. Clearly for each $x \in F^{\#}$, $C_V(x)/C_V(F)$ is a one-dimensional $GF(4)$-subspace. Since there are only five $GF(4)$ 1-spaces in $W/C_V(F)$, $C_V(x) = C_V(y)$ for some pair $x, y \in F^{\#}$, which of course contradicts (iii). Thus (vi) is true.

Next, assume that $F \le S_1 \in Syl_2(G)$, $|F| = 2^4$, and F is not normal in S_1. Then there exists $s \in S_1$ with $F \ne F^s$. If $|F \cap F^s| \ge 4$, then (iii) implies that $C_V(F \cap F^s) = C_V(F) = C_V(F^s)$ and that $F = C_G(C_V(F)) = C_G(C_V(F^s)) = F^s$, which is absurd. Therefore, since $|S_1| = 2^7$, $|F \cap F^s| = 2$ and $S_1 = FF^s$. Hence $C_V(F) \cap C_V(F^s) \le C_V(S_1)$. Because $C_V(F \cap F^s) \cong E(2^8)$ and $[C_V(F \cap F^s) : C_V(F)] = 2^2 = [C_V(F \cap F^s) : C_V(F^s)]$, $|C_V(F) \cap C_V(F^s)| \ge 2^4$. However, $|C_V(S_1)| = 2^2$ and so we have a contradiction.

Finally assume that $H = 3\,\mathrm{Aut}(M_{22})$ and U is a faithful $GF(2)H$-module which when restricted to $F^*(H)$ is an irreducible quadratic $GF(2)F^*(H)$-module. Then, by Theorem 2.1, U has dimension 12 and, for any $t \in H \setminus F^*(H)$ of order 2, $\langle t, Z(F^*(H))\rangle \cong S_3$. Since $Z(F^*(H))$ operates fixed-point freely on U, we deduce $[U : C_U(t)] = 2^6$ as stated in (viii). This completes the proof of Proposition 2.12.

PROPOSITION 2.13. *Suppose that $G \cong J_2$, Co_1, or $3Suz$, V is a nontrivial irreducible quadratic module for G, and F is a maximal quadratic subgroup of G. Then the following hold*:

(i) $|F| = 4$ *and all the involutions of F are either of class $2A$ or $2B$.*

(ii) *if $G \cong J_2$, then $[V : C_V(t)] = 2^4$ if t is of class $2A$, and $[V : C_V(t)] \ge 2^6$ if t is of class $2B$.*

(iii) *If $G \cong Co_1$, then*

(a) $[V : C_V(t)] = 2^8$, *if t is of class $2A$;*

(b) $[V : C_V(t)] = 2^{12}$, *if t is of class $2B$;*

(c) *for $F \lhd S$, where $S \in Syl_2(G)$, $[V, C_S(F)]$ has codimension 1 in V.*

(iv) *If $G \cong 3Suz$, then $[V : C_V(t)] = 2^8$ if t is of class $2A$, and $[V : C_V(t)] \ge 2^6$ if t is of class $2B$.*

(v) $[V : C_V(F)] \ge 2^5$ *and* $\|[V, F]\| \ge 2^5$.

Proof. By [MS, Theorem 2], $|F| = 4$ and all the involutions of F are conjugate under the action of $N_G(F)$. So, by [$\mathbb{A}$], F contains only involutions of class $2A$ or $2B$. This verifies (i).

Suppose that t is of class $2A$. Then for $G \cong J_2$, Co_1, or $3Suz$, we have from [MS, Lemmas 3.6, 3.7, and 3.9], respectively, $[V : C_V(t)] = 2^4$, 2^8, or

2^8. Next assume that t has class $2B$. In the case that $G \cong J_2$ [MS, Theorem 3] implies that $[V : C_V(t)] \geq 2^6$, for t an involution in $C_G(2A) \setminus O_2(C_G(2A))$ (such t exist since $C_G(2A) \cong 2_-^{1+4} : A_5$). Since $[V : C_V(s)] = 2^4$ for s of class $2A$, we deduce that $[V : C_V(t)] \geq 2^6$ for t of class $2B$, which proves (ii).

Next we assume that $G \cong Co_1$ (respectively $3Suz$) and that t is of class $2B$ with $[V : C_V(t)] \leq 2^{11}$ (respectively, $\leq 2^5$). Then, by [$\mathbb{A}$], $C_G(t) \cong (2^2 \times G_2(4)) : 2$ (respectively, $\cong (2^2 \times SL_3(4)) : 2$) and $O^2(C_G(t)) \cong G_2(4)$ (respectively, $\cong SL_3(4)$); in particular, as the minimal nontrivial $GF(2)$-representation of $G_2(4)$ is 12-dimensional and the minimal nontrivial $GF(2)$-representation of $SL_3(4)$ is 6-dimensional, we observe that $O^2(C_G(t))$ centralizes $V/C_V(t)$. Hence, by Lemma 2.5, $\langle t, t_1 \rangle$ is quadratic for all involutions t_1 in $O^2(C_G(t))$. Thus [MS, Theorem 3] implies that every involution of $\langle t \rangle \times O^2(C_G(t))$ is of class $2B$. Now if $G \cong 3Suz$, then $|\langle t \rangle \times O^2(C_G(t))|_2 = 2^7$ and, using the notation of [MS, Theorem 3] and [$\mathbb{A}$], $Z(O_2(P_2)) \cap (\langle t \rangle \times O^2(C_G(t))) = 1$. Since a Sylow 2-subgroup of $SL_3(4)$ is isomorphic to a Sylow 2-subgroup of $2^4 : A_5$, a Sylow 2-subgroup of $\langle t \rangle \times O^2(C_G(t))$ is generated by involutions. Thus there are involutions from $\langle t \rangle \times 0^2(C_G(t))$ in P_2 and not in $O_2(P_2)$. But then, using [MS, Theorem 3] again, $[V : C_V(t)] \geq 2^8$, which is a contradiction. This completes the proof of (iv). So we now assume that $G \cong Co_1$. Then since $|O^2(C_G(t))|_2 = 2^{12}$ and $|M_{24}|_2 = 2^{10}$, we may choose P a subgroup of G with $P \cong 2^{11}M_{24}$ such that $O^2(C_G(t)) \cap O_2(P) \neq 1$. But [$\mathbb{A}$] tells us that $O_2(P)$ contains only involutions of class $2A$ and $2C$, so we have a contradiction here as well. Hence (iii)(b) holds.

Now we suppose that F is a maximal quadratic subgroup of G and that $[V : C_V(F)] = 2^4$ or $\|[V, F]\| = 2^4$. Then (ii), (iii), and (iv) imply that $G \cong J_2$ and that F contains only involutions t of class $2A$ with $C_V(t) = C_V(F)$, respectively, $[V, t] = [V, F]$. But this then forces $C_V(F)$, respectively, $[V, F]$, to be invariant under the action of $\langle C_G(t), N_G(F) \rangle = G$ (where $t \in F^\#$), which is a contradiction. Hence (v) holds.

It remains to prove (ii)(c). Select $P_1 \geq S$ with P_1 of shape $2^{11}M_{24}$ and $Q_1 = O_2(P_1)$. Since $E(2^2) \cong F \lhd S$, $\|[F, Q_1]\| \leq 2$. Now Proposition 2.10(ii) and (iii) (applied to the action of P_1/Q_1 on Q_1) implies that $F \leq Q_1$. Thus $Q_1 \leq C_S(F)$, so $[V, C_S(F)] \geq [V, Q_1]$ and [MS, Lemma 3.7(b)] gives (iii)(c).

PROPOSITION 2.14. *Suppose that $G \cong Co_2$, $S \in Syl_2(G)$, and V is a nontrivial irreducible quadratic module for G. Let F be a maximal quadratic subgroup of G. Then*

(i) $|F| = 4$ *and* $[V : C_V(t)] \geq 2^6$ *for each* $t \in F^\#$;

(ii) *suppose that* $F \lhd S$; *then* $[V, C_S(F)]$ *has codimension* 1 *in* V.

Proof. From [MS, Theorem 2], $|F| = 4$ and F contains one element of class $2A$ and two of class $2B$ (see the beginning of the proof of [MS, Lemma 3.8]). Now, by [MS, Theorem 3] and using $V/C_V(t) \cong [V,t]$ as $GF(2)C_G(t)$-modules, the $2A$ elements have centralizer index at least 2^6. Now assume that t is a $2B$ element. We refer to [MS, Lemma 3.8(b)] where the restriction of V to $P = C_G(t)$ is given (wrongly):

$$O \underset{\bar{4}}{<} C_V(Q) \underset{4}{<} C_V(Z(Q)) \underset{6}{<} [V, Z(Q)] \underset{\bar{4}}{<} [V,Q] \underset{4}{<} V.$$

Hence, as $V/C_V(t) \cong [V,t]$ as a $GF(2)C_G(t)$-module, the centralizer index of an element of class $2B$ is 2^8, and this completes (i).

Let $G \geq P_1 \geq S$ with P_1 of shape $2^{10}\,\mathrm{Aut}(M_{22})$ and $Q_1 = O_2(P_1)$. Since $E(2^2) \cong F \lhd S$, $\|F, Q_1\| \leq 2$. Now, Proposition 2.11 (applied to the action of P_1/Q_1 on Q_1) implies that $F \leq Q_1$. Thus $Q_1 \leq C_S(F)$, whence $[V, C_S(F)] \geq [V, Q_1]$ and [MS, Lemma 3.8(b)] gives (ii).

For quick and easy reference, we summarize the outcome of the six previous propositions.

THEOREM 2.15. *Suppose that $G \in \mathsf{S}$, $S \in Syl_2(G)$, and that V is a nontrivial faithful irreducible quadratic module for G. Let F be a maximal quadratic subgroup of G and suppose that $t \in F^*$. Then*

(i) $|F| = 4$, *or* $F^*(G) \cong 3M_{22}$, $|F| = 2^3$ *or* 2^4, *and* $F \leq F^*(G)$;

(ii) $[V : C_V(F)] \geq 2^5$ *and* $[V : C_V(t)] \geq 2^4$, *or* $G \cong \mathrm{Aut}(M_{22})$, $[V : C_V(F)] \geq 2^4$ *and* $[V : C_V(t)] \geq 2^3$;

(iii) $[V : C_V(F_1)] > |F_1|$ *for all quadratic* $F_1 \neq 1$; *in particular, V is not FF*;

(iv) $|C_V(S)| = |V/[V,S]| = 2$ *or* $G \cong 3M_{22}$, J_2, *or* $3Suz$ *and* $|C_V(S)| = |V/[V,S]| = 2^2$;

(v) *For any irreducible $GF(2)G$-module W and any involution $s \in G$, either* $[W : C_W(s)] \geq 2^4$ *or* $G \cong \mathrm{Aut}(M_{22})$, *$W$ is a quadratic module,* $[W : C_W(s)] = 2^3$, *and s is contained in a quadratic fours group on W.*

Proof. Parts (i) and (ii) are a collection of the results from Theorem 2.1 and Propositions 2.9–2.14. While if $|F_1| \geq 4$, then (iii) follows from (i) and (ii). Thus to complete the proof of (iii) we need to explore the case when $F_1 = \langle t \rangle$ has order 2 and $[V : C_V(t)] = 2$. Using results of Propositions 2.9–2.14 we observe that t is not contained in any quadratic fours group. Thus, because $\|[V,t]\| = 2$, Lemma 2.5 implies that t is the only involution in $C_G(t)$. Hence, t is a 2-central involution and $S \in Syl_2(G)$ is either cyclic or generalized quaternion. This is not the case for any group in S. Therefore, (iii) holds.

Next, (iv) follows from [MS, part b) of Lemmas 3.2−3.9].

Finally, Propositions 2.7 and 2.8 together with part (ii) and Proposition 2.11(iv) give (v).

PROPOSITION 2.16. *Suppose that $G \in \mathsf{S}$, $S \in Syl_2(G)$, M is a maximal subgroup of S, $z \in \Omega_1(Z(M))$, and V is a nontrivial irreducible $GF(2)G$-module. Then $[V, M, z] \neq 0$. In particular, we have* (i) $[V, S, z] \neq 0$ *and* (ii) $[V, S, t] \neq 0$ *for all involutions $t \in G$.*

Proof. Let G, S, M, and z be as in the statement of the proposition and assume that $[V, M, z] = 0$. Suppose that $t \in M$ is an involution of M other than z. Then, by Lemma 2.5(i), $F = \langle t, z\rangle$ is quadratic on V. Moreover, using the three subgroup lemma, we have $[V, z, M] = 0$ and so $[V, z] \leq C_V(M)$. Now S acts as an involution on $C_V(M)$ and thus, employing Theorem 2.15(iv),

$$|[V, z]| \leq |C_V(M)| \leq 2|C_V(S)| \leq 2^3.$$

Then $G \cong \mathrm{Aut}(M_{22})$ by Theorem 2.15(ii). Now Theorem 2.15(iv) gives $|C_V(S)| = 2$, which gives the untenable $|[V, z]| \leq 2^2$. Hence we conclude that M contains no involutions other than z, so $z \in Z(S)$. Therefore, M is cyclic or generalized quaternion. Moreover, this also yields $m(S) \leq 2$. Using [$\mathbb{A}$] we get that $G \cong M_{11}$ and $M \cong Q_8$. Now M_{11} contains one class of elements of order 2 and one class of elements of order 4. Further M_{11} has a subgroup isomorphic to a Frobenius group of order 20. Now choose x of order 4 in M and let $Y \cong C_5$ be normalized by x. We have $x^2 = z$ and $[V, x^2, x] \leq [V, z, M] = 0$. By [MS, Lemma 1.4] $C_Y(x^2) \neq 1$, which contradicts the structure of the Frobenius group. Hence $[V, M, z] \neq 0$. Statement (i) now follows immediately.

For part (ii), suppose that $[V, S, t] = 0$ and put $T = \langle t\rangle$. Then we have from (i) that $T \cap \Omega_1(Z(S)) = 1$; thus, by Lemma 2.5(ii), $\langle T, \Omega_1(Z(S))\rangle$ acts quadratically on V. In particular, V is a quadratic G-module. But then Lemma 2.15(iv) implies that $[V : [V, S]] \leq 2^2$ and so $[V : C_V(T)] \leq 2^2$, which is at odds with Lemma 2.15(v).

LEMMA 2.17. *Suppose that $G \cong J_1$ or Fi_{23}, t is an involution in G, and V is a nontrivial $GF(2)G$-module.*

(i) *If $G \cong J_1$, then* $\dim_{GF(2)} V \geq 18$ *and* $[V : C_V(t)] \geq 2^9$.

(ii) *If $G \cong Fi_{23}$, then either t is of class $2A$ and $[V : C_V(t)] \geq 2^{54}$ or $[V : C_V(t)] \geq 2^{14}$.*

Proof. Suppose first that $G \cong J_1$. Then, as there is exactly one class of involutions in G and an element of order 19 is inverted, (i) holds.

Assume now that $G \cong Fi_{23}$, t is of class $2A$, t_1 is of class $2B$ or $2C$, and $S \in Syl_2(G)$. Then $C_G(t) = O^2(C_G(t)) \cong 2Fi_{22}$. If $\eta(C_G(t), V/C_V(t)) = 0$, then $[V, C_G(t)] \leq C_V(t)$. Hence, using Lemma 2.5(i) and the fact that $C_G(t) \backslash \langle t \rangle$ contains involutions we have a contradiction to Theorem 2.1. Therefore, $\eta(C_G(t), V/C_V(t)) \geq 1$. This makes $V/C_V(t)$ into a nontrivial $GF(2)Fi_{22}$-module and, as Fi_{22} contains an extraspecial 3-subgroup of order 3^7, [A, 7.1] implies that $[V : C_V(t)] \geq 2^{2.3^3} = 2^{54}$. Now, without loss of generality we may assume that $t_1 \in \Omega_1(Z(S)) \cong E(2^2)$ and so in $H = C_G(t)/\langle t \rangle \cong Fi_{22}$, t_1 projects to an involution s of class $2B$ with $C_H(s) \cong (2^1 \times 2^{1+8} : U_4(2)) : 2$. Assume that W is a nontrivial $GF(2)H$-module. Set $X := O^2(C_H(2))$. Then using the Atlas and Lemma 2.2, $[H : X] \leq 4$, $|O_2(X)| \geq 2^9$, and $m(S/\langle t \rangle) = 10$. Therefore, $O_2(X) \backslash \langle s \rangle$ contains involutions. From Lemma 2.5(i) and Theorem 2.1 we see that $O_2(X)$ does not centralize $\overline{W} := W/C_W(s)$ and arguing as above we get $\eta(X, \overline{W}) \geq 1$. Thus at least one of $\eta(X, [\overline{W}, O_2(X)]) \geq 1$ or $\eta(X, \overline{W}/[\overline{W}, O_2(X)]) \geq 1$ holds. Hence, as X contains an extraspecial 3-subgroup of order 3^3 we get from [A, 7.1] that either $|\overline{W}/[\overline{W}, O_2(X)]| \geq 2^{2.3} = 2^6$ or $|[\overline{W}, O_2(X)]| \geq 2^6$. Thus as $\overline{W} > [\overline{W}, O_2(X)] \geq 0$, $[W : C_W(s)] \geq 2^7$. Using this fact in tandem with $\eta(C_G(t), V) \geq 2$ gives $[V : C_V(t_1)] \geq 2^{14}$ and the proof of (ii) is complete.

The following three lemmas are deployed in Section 7 to help analyse the $b = 1$ case.

LEMMA 2.18. *Suppose that $G \in \mathsf{S}$, $S \in Syl_2(G)$, $1 \neq A \leq S$, and V is a faithful irreducible $GF(2)G$-module. If V is an $(FF + 1)$-module with offending elementary abelian 2-subgroup A, then $[V : C_V(A)] = 2|A|$ and the pair (G, V) is one of $(M_{24}, Todd)$, $(M_{24}$, Golay code$)$, $(M_{23}, Todd)$, $(M_{22}, Todd)$, $(\mathrm{Aut}(M_{22}), Todd)$, or $(\mathrm{Aut}(M_{22})$, Golay code$)$. Furthermore, we also have the following*:

(i) *Suppose that $G \cong M_{24}$ and put $P_1 \sim 2^6 3S_6$, $P_2 \sim 2^6(L_3(2) \times S_3)$, $P_3 \sim 2^4 A_8$, and $Q_i = O_2(P_i)$, $i = 1, 2, 3$. Then* (a) *V is the Todd module and $A \in \{Q_3^G\}$ or* (b) *V is the Golay code module and $A \in \{Q_1^G\}$.*

(ii) *Suppose that $G \cong M_{23}$ and put $P \sim 2^4 A_7$ and $Q = O_2(P)$. Then $A \in \{Q^G\}$.*

(iii) *Suppose that $G \cong \mathrm{Aut}(M_{22})$ and put $P_1 \sim 2^4 S_6$, $P_2 \sim 2^{4+1} S_5$, and $Q_i = O_2(P_i)$, $i = 1, 2$. Then either* (a) *V is the Todd module and $A \in \{Q_1^G\}$ or* (b) *V is the Golay code module and $A \in \{Q_2^G\}$.*

Proof. By [A, Theorem 1], $F^*(G) \cong M_{22}$, M_{23}, or M_{24} and $[V : C_V(A)] = 2|A|$, while [A, 11.4 and 11.5] implies that (G, V) is isomorphic to one of the pairs listed in the statement of the lemma. Suppose first that $G \cong M_{24}$. Then V is either the Todd module or the Golay code module. In particular, V is a quadratic G-module and so Proposition 2.10 applies. Suppose that A is an $FF+1$ offender. From $[V:C_V(A)] = 2|A|$ and Proposition 2.10(ii) and (iii), $|A| \geq 2^3$. If $|A| = 2^3$, then $[V:C_V(A)] = 2^4$ which, using Proposition 2.10(ii) and (iii), implies that $C_V(A) = C_V(t)$ for all $t \in A^{\#}$. But then A acts quadratically on V, contradicting Proposition 2.10 and so $|A| \neq 2^3$. Thus by Lemma 2.2, $|A| \in \{2^4, 2^5, 2^6\}$.

Assuming that $|A| = 2^4$ we aim to show that (i)(a) holds. Thus assume that $A \notin \{Q_3^G\}$. Suppose that A contains only $2B$ elements. Then Proposition 2.10(iii) implies that $C_V(A) = C_V(t)$ for all $t \in A^{\#}$ and so A is a quadratic subgroup of G, which is against Proposition 2.10(i). Therefore, we may assume that $t \in A^{\#}$ is a $2A$ element. So after replacing A and S by conjugates if necessary we may assume that $Z := Z(S) \leq A \leq S$; in particular, $A \cap Q_3 \neq 1 \neq A \cap Q_1$. Employing Proposition 2.10(ii) we see that $[C_V(Z):C_V(A)] = 2$ and so

(2.18.1) $\quad [C_V(Q_i):C_{C_V(Q_i)}(A)] \leq 2$ for $i = 1, 3$.

If V is the Todd module, then, by [MS, Lemma 3.5], $C_V(Q_3)$ is isomorphic to the six-dimensional A_8 permutation module, which does not admit transvections. Therefore, as $A \nleq Q_3$, (2.18.1) implies that V is the Golay code module. Thus, by [MS, Lemma 3.5], $C_V(Q_1)$ is isomorphic to a natural S_6-module. So, again using (2.18.1), $|AQ_1/Q_1| \leq 2$ and $|A \cap Q_1| \geq 2^3$. Now, if $A \cap Q_1$ contains a $2B$ element t then we have $C_V(Q_1) \leq C_V(t) = C_V(A)$ and we get $A \leq Q_1$. Assuming that $A \cap Q_1$ contains a fours group E which contains only $2A$ elements we either get $C_V(Q_1) \leq C_V(E) = C_V(A)$ and $A \leq Q_1$, or $C_V(E) = C_V(e)$ for each $e \in E^{\#}$. In the latter case E is quadratic and $[V:C_V(E)] = 2^4$, which is against Proposition 2.10(iv). Therefore we infer that $A \leq Q_1$. Set $P_0 = C_G(Z)$ and $Q_0 = O_2(P_0)$. Now, $\eta(P_0, C_V(Z)) = 2$, and so $[C_V(Z):C_V(A)] = 2$ implies that $A \leq Q_0$. Then $A = Q_1 \cap Q_0$ and $C_V(A)$ is invariant under $P_1 \cap P_0 \sim 2^9 S_3$. If A contains a $2B$ element t then $C_V(A) = C_V(t)$ is invariant under $C_{P_1}(t) \sim 2^6 S_5$ as well, which forces $C_V(A)$ to be invariant under P_1, which is against [MS, Lemma 3.5]. Hence A contains only $2A$ elements. So

(2.18.2) $\quad A = Q_1 \cap O_2(C_G(t))$ is normal in $C_G(t) \cap P_1$ for all $t \in A^{\#}$.

With the help of (2.18.2) we can now deduce a contradiction. We know that $P_1 \cap P_0 \sim 2^9 S_3$ is a maximal subgroup of P_1. Thus, as A is not normal in P_1, (2.18.2) implies $C_G(t) \cap P_1 = P_1 \cap P_0$ for all $t \in A^{\#}$; but then $A \leq Z(S)$, which is absurd. With this contradiction we conclude that $A \in \{Q_3^G\}$ and (i)(a) holds by [MS, Lemma 3.5].

Next assuming that $|A| = 2^5$, we aim for a contradiction. Evidently $A \nleq Q_3$ and $A \cap Q_3 \neq 1$. As $Q_3^{\#}$ contains only $2A$ elements we may assume $A \geq Z$. Suppose first that V is the Todd module. If $|A \cap Q_3| \geq 2^2$, then we select a fours group $E \leq A \cap Q_3$ and as above we find that $[V:C_V(E)] \geq 2^5$. But then $[C_V(E):C_V(A)] \leq 2$ and so $[C_V(Q_3):$

$C_{C_V(Q_3)}(A)] \leq 2$, which again forces A to act as a transvection on the six-dimensional A_8 permutation module, which is absurd. Thus $A \cap Q_3 = Z$ and $|AQ_3/Q_3| = 2^4$, and in this case calculation on the six-dimensional A_8 permutation module shows that $C_{C_V(Q_3)}(A)$ has order 2^2, a fact which is at variance with $[C_V(Z) : C_V(A)] = 2^2$, which forces $[C_V(Q_3) : C_{C_V(Q_3)}(A)] \leq 2^2$. Therefore V is the Golay code module. Because $S/Q_1 \cong D_8 \times 2$, $|Q_1 \cap A| \geq 2^2$. Then, as above, A operates as a transvection on $C_V(Q_1)$ and so $|A \cap Q_1| \geq 2^4$. If $C_V(A \cap Q_1) > C_V(A)$, then $A \cap Q_1$ is an $FF+1$ offender on V and by (i)(a) V is the Todd module, a fact which we have disproven. Hence $C_V(Q_1) \leq C_V(A \cap Q_1) = C_V(A)$ and so $A \leq Q_1$. Now, $\eta(P_0, C_V(Z)) = 2$, and as $V/C_V(Z) \cong [V, Z]$ as P_0-modules the noncentral chief factors in $C_V(Z)$ are not isomorphic $P_0/Q_0 \cong L_3(2)$-modules. Note that $A \nleq Q_0$ as $|Q_0 \cap Q_1| = 2^4$ and $A \leq Q_1$. Therefore, since $[C_V(Z) : C_V(A)] = 2^2$, we must have $|AQ_0/Q_0| = 2$. Thus $|A \cap Q_0| = 2^4$ and, because $A \leq Q_1$, $A \cap Q_0 = Q_1 \cap Q_0$. In particular, $A \cap Q_0 \triangleleft P_0 \cap P_1 \sim 2^9 S_3$. Now, by (i)(a), $C_V(A \cap Q_0) = C_V(A)$ is invariant under $P_0 \cap P_1$ and so $C_V(A) = C_V(\langle A^{P_0 \cap P_1}\rangle) = C_V(Q_1)$, and this is against [MS, Lemma 3.5]. Thus we have show that $|A| \neq 2^5$.

Lastly, for this case, assume that $|A| = 2^6$. Then $A \in \{Q_1^G\} \cup \{Q_2^G\}$. Using [MS, Lemma 3.5] then immediately gives $A \in \{Q_1^G\}$, and (i)(b) holds.

Assuming that $G \cong M_{23}$, we note that the M_{23} Todd module is the restriction of the M_{24} Todd module. Thus, any offender in M_{23} fuses in M_{24} to a subgroup in $\{Q_3^G\}$. So (ii) holds.

Next suppose that $G \cong \mathrm{Aut}(M_{22})$ and set $H = F^*(G)$. As in the M_{24} case, V is a quadratic module for $\mathrm{Aut}(M_{22})$; thus we may use Proposition 2.11. If $|A| = 2^2$, then as $A \cap H \neq 1$, Proposition 2.11(iii) delivers a contradiction. So Proposition 2.11(iv) and Lemma 2.2 imply $|A| \in \{2^3, 2^4, 2^5\}$. Assume that $|A| = 2^3$. Then $A_0 = A \cap H$ has order at least 2^2 and $C_V(A_0) = C_V(A) = C_V(t)$ for all $t \in A_0^{\#}$ by Proposition 2.11(iii). But then A_0 is a quadratic fours group in H, which is prohibited by Proposition 2.11(ii). Therefore, $|A| \geq 2^4$.

Assuming now that $|A| = 2^4$, we aim to prove (iii)(a). As H contains a unique class of involutions $(2A)$ we may assume that $A \geq Z := Z(S)$. Suppose first that V is the Todd module. Then Proposition 2.11 implies that $[C_V(Z) : C_V(A)] = 2$. Assume that $A \nleq Q_1$. Then $[C_V(Q_1) : C_{C_V(Q_1)}(A)] \leq 2$ and so, using [MS, Lemma 3.3], $|AQ_1/Q_1| = 2$ and $|A \cap Q_1| = 2^3$. As $C_V(A) \ngeq C_V(Q_1)$, we have $C_V(A \cap Q_1) > C_V(A)$, which is to say $A \cap Q_1$ is an $FF+1$ offender; this being already contradicted we have a contradiction to the assertion $A \nleq Q_1$. Hence (iii)(a) holds then V is the Todd module. Now assume that V is the Golay code module. Assume that $A \nleq Q_2$. Then as above A operates as a transvection on $C_V(Q_2)$, which is an orthogonal S_5-module. This implies that $|A \cap Q_2| =$

2^3. Again $C_V(A \cap Q_2) = C_V(A)$ or $A \cap Q_2$ is an $FF + 1$ offender, a situation already shown to be false. Thus $C_V(A) = C_V(A \cap Q_2) \geq C_V(Q_2)$ and hence $A \leq Q_2$. Set $R_2 = H \cap Q_2$. Then $E(2^4) \cong R_2 \lhd P_2$. If $A = R_2$, then $C_V(R_2)$ has dimension 5 and contains $C_V(Q_2)$, which is isomorphic to an orthogonal $S_5 \cong (P_2 \cap H)/R_2$-module. But then $C_V(R_2)$ splits as a $(P_2 \cap H)R_2$-module which then gives the impossible $C_V(R_2) = C_V(Q_2)$. So we conclude that $A \neq R_2$.

Because Q_2 is an indecomposable P_2/Q_2-module and $P_2/Q_2 \cong S_5$ has six Sylow 5-subgroups and ten Sylow 3-subgroups we deduce that $Q_2 \setminus R_2$ consists of six $2C$ elements and ten $2B$ elements. Consequently there exists $t \in A$ with t of class $2B$. Put $P = C_G(t)$ $(\cong 2^{1+3}L_3(2))$ and $Q = O_2(P)$. From [MS, Lemma 3.3], $A \neq Q$. By Proposition 2.11(ii) and (iv) and [MS, Lemma 3.3], $\eta(P, C_V(t)) = 2$ with the noncentral P/Q-chief factors in $C_V(t)$ being nonisomorphic. Thus, because $[C_V(t) : C_V(A)] = 2^2$, $|AQ/Q| = 2$ and hence $|A \cap Q| = 2^3$. Finally, as $\eta(P, C_V(Q)) = 1$, $C_V(Q) \nleq C_V(A)$ and so $C_V(A)C_V(Q)$ is a subspace of codimension at most 4 centralized by $A \cap Q$ of order 2^3, which has been ruled out.

Lastly assume that $|A| = 2^5$. Then, as Q_2 is not an FF-module for S_5, $A \in \{Q_2^G\}$ and (iii)(b) follows using [MS, Lemma 3.3].

LEMMA 2.19. *Assume that $G \in \{M_{23}, M_{24}\}$, $S \in Syl_2(G)$, and W is a $GF(2)G$-module with $W > [W, G] = V$, where V is irreducible of dimension 11 and $C_W(G) = 0$. Then $G \cong M_{24}$, V is the Todd-module and $\dim_{GF(2)} W = 12$. Moreover, if we select P_2 and P_3 subgroups of G $(\cong M_{24})$ containing S with $P_2 \sim 2^{3.2}(L_3(2) \times S_3)$ and $P_3 \sim 2^4 A_8$ and set $Q_i = O_2(P_i)$, $i = 2, 3$, then $[W : C_W(Q_2)] = 2^9$ and $[W : C_W(Q_3)] = 2^5$; in particular, Q_3 is an $FF + 1$ offender on W.*

Proof. Since $V = [W, G] < W$ we can pick a G-submodule W_1 with $V \leq W_1 \leq W$ and $\dim_{GF(2)} W_1 = 12$. We prove the result for W_1 and then show that $W_1 = W$. Observe that by Gaschütz's theorem [Hu, 17.4] and $C_W(G) = 0$, $C_V(S) = C_{W_1}(S)$.

Let $x \in G$ be an element of order 23. Then $W_1 = C_{W_1}(x) \oplus [W_1, x]$ and $C_{W_1}(x) \setminus V$ is nonempty. Select $v \in C_{W_1}(x) \setminus V$ and put $T := Stab_G(v)$. Then $|\{v^G\}| = [G : T] \leq 2^{11} = 2048$. Now $23 \mid |T|$, so utilizing the Atlas [$\mathbb{A}$] we see that if $G \cong M_{23}$, there is a unique maximal subgroup which has order divisible by 23 and it has index 40,320 > 2048. Thus if $G \cong M_{23}$, then $C_{W_1}(G) \neq 0$, a contradiction. So we deduce that $G \cong M_{24}$. Again consulting [$\mathbb{A}$], in this case there are exactly two maximal subgroups which have order divisible by 23. Arguing as in the last case, we find that the only possibility is that $T \cong M_{23}$. Let R_3 be the maximal subgroup of T with shape $2^4 A_7$.

Then (after conjugating if needed) $O_2(R_3) = Q_3$. Thus $[C_{W_1}(Q_3): C_V(Q_3)] = 2$. Hence, if V is the Golay code module, [MS, Theorem 3] implies that $|C_{W_1}(Q_3)| = 2^2$ and, as P_3 is perfect, $C_{W_1}(Q_3) = C_{W_1}(P_3) \leq C_{W_1}(S)$, which is a contradiction. Therefore, V is the Todd-module and $[W_1 : C_{W_1}(Q_3)] = 2^5$. Next suppose that $C_{W_1}(Q_2) > C_V(Q_2)$. By [MS, Lemma 3.5], $C_V(Q_2)$ is an irreducible three-dimensional space for $O^3(P_2)$, so $|C_{W_1}(Q_2)| = 2^4$. Now $Q_2Q_3 = O_2(O^3(P_2))$, whence Q_3 induces an involution on $C_{W_1}(Q_2)$ and $[C_{W_1}(Q_2), Q_3]$ is a $O^3(P_2)$-invariant subspace of $C_V(Q_2)$ of order at most 4. We conclude that $C_{W_1}(Q_2) \leq C_{W_1}(Q_3)$. Now, by [MS, Lemma 3.5] again, $C_V(Q_3)$ is an irreducible six-dimensional $GF(2)P_3$-module and $C_{W_1}(Q_3)$ is a nonsplit extension of $C_V(Q_3)$ as a $GF(2)A_8$-module. We claim that P_3 has orbits of length 8 and 56, with stabilizers $T_1 = 2^4A_7$ and $T_2 = 2^4(A_5 \times 3):2$, on $C_{W_1}(Q_3)\backslash C_V(Q_3)$. Clearly we only need to consider the action of A_8 on $C_{W_1}(Q_3)\backslash C_V(Q_3)$. Let $F_1 \leq A_8$ with F_1 a Frobenius group of order 21. Then $F_1 \leq Stab(v_1)$ for some $v_1 \in C_{W_1}(Q_3)\backslash C_V(Q_3)$. By [$\mathbb{A}$], $Stab(v_1)$ is contained in a maximal subgroup of A_8 isomorphic to either A_7 or $2^3L_3(2)$. Since $|C_{W_1}(Q_3)\backslash C_V(Q_3)| = 64$ we see that $Stab(v_1) \cong A_7$ or $2^3L_3(2)$ are the only possibilities, with the latter ruled out by $C_W(S_1) = C_V(S) \leq V$. So $Stab(v_1) \cong A_7$. Choosing $F_2 \leq A_8$ with F_2 cyclic of order 15 a similar argument gives that $Stab(v_2) \cong (A_5 \times 3):2$ for some $v_2 \in C_{W_1}(Q_3)\backslash C_V(Q_3)$. However, recalling that $Q_2Q_3/Q_3 \cong E(2^3)$, we see that Q_2Q_3/Q_3 cannot be conjugate to a subgroup of either A_7 or $(A_5 \times 3):2$ (as $m_2(A_7) = 2 = m_2((A_5 \times 3):2)$). Thus we have a contradiction to our supposition that $C_W(Q_2) > C_V(Q_2)$. Therefore $[W_1 : C_{W_1}(Q_2)] = 2^9$.

Finally assume that $W > W_1$. Then, as G is simple, W/V is a direct sum of trivial modules. Therefore there exists a G-submodule of dimension 12 with $W_2 > V$ and $\dim_{GF(2)}(W_1W_2) = 13$. Applying the results above to W_2, we find that $C_{W_1W_2}(Q_3)$ has dimension 8. Now, by [Be], $H^1(A_8, 6) \cong GF(2)$ and so $|C_{W_1W_2}(S)| \geq 2^2$, a contradiction. Therefore we conclude that $W_1 = W$ and this completes the proof of Lemma 2.19.

Lemma 2.20. *Assume that* $G \cong M_{22}$ (*respectively*, $\mathrm{Aut}(M_{22})$), $S \in Syl_2(G)$, $P_1 \sim 2^4A_6$ (*respectively*, 2^4S_6), $P_2 \sim 2^4S_5$ (*respectively*, $2^{4+1}S_5$), $Q_i = O_2(P_i)$, $i = 1, 2$, *and W is a $GF(2)G$-module with* $W > [W, G] = V$, *where V is irreducible of dimension* 10 *and* $C_W(G) = 0$. *Then* $[W : C_W(Q_1)] \geq 2^6$ *and* $[W : C_W(Q_2)] \geq 2^7$. *In particular, Q_1 and Q_2 are not $FF + 1$ offenders on W.*

Proof. It suffices to prove the lemma for a G-submodule W_1, where W_1 has dimension 11 and $W_1 > V$. Assume first that V is the 10-dimensional Golay code module. Then $[V : C_V(Q_2)] = 2^6$ and (using [MS, Lemma 3.3]) $C_V(Q_2)$ is an orthogonal S_5-module. If $C_{W_1}(Q_2) > C_V(Q_2)$, then (as the orthogonal S_5-module is projective) $|C_V(S)| \geq 2^2$ and so $C_{W_1}(G) > 0$, by Gaschütz's theorem [Hu, 7.14], which is a contradiction. Therefore, $[W_1 : C_{W_1}(Q_2)] \geq 2^7$ and, by [MS, Lemma 3.3], $[W_1 : C_{W_1}(Q_1)] \geq 2^9$. So the result holds in this case.

Now we assume that V is the 10-dimensional Todd module. In this case $[V : C_V(Q_1)] = 2^5$, again, if $C_{W_1}(Q_1) > C_V(Q_1)$, then $C_{W_1}(Q_1)$ is an A_6-module (respectively, S_6-module) of dimension 6 with (using [MS, Lemma 3.3]) $[C_{W_1}(Q_1), O^2(P_1)]$ of dimension 4. There are no such indecomposable P_1/Q_1-modules and so once more we obtain $|C_{W_1}(S)| \geq 2^2$, which is again a contradiction. Hence, $[W_1 : C_{W_1}(Q_1)] = 2^6$ and, by [MS, Lemma 3.3], $[W_1 : C_{W_1}(Q_2)] \geq 2^9$ and the lemma is proven.

To finish with the sporadic groups in this section we present two structural lemmas.

LEMMA 2.21. *Suppose that $G \in \mathsf{S}$ and $S \in Syl_2(G)$. Then one of the following holds*:

(i) $|\mathrm{Aut}(S)|_{2'} = 1$;

(ii) *$G/Z(G) \cong J_2$, J_3, Suz, or HN and an odd automorphism of S has order dividing* 3;

(iii) *$G \cong J_1$ and* $\mathrm{Aut}(S) \cong GL_3(2)$.

Proof. See [PR4].

LEMMA 2.22. *Suppose that $G \in \mathsf{S}$ and $T \in Syl_2(G)$. Put $S = T \cap O^2(G)$. Then one of the following holds*:

(i) $\Omega_1(Z(T)) = \Omega_1(Z(S))$ *has order* 2.

(ii) $G = O^2(G) \cong J_1$ *and* $|\Omega_1(Z(S))| = 2^3$.

(iii) $G = O^2(G) \cong Fi_{23}$ *and* $|\Omega_1(Z(S))| = 2^2$.

Proof. Clearly there is no loss in assuming that $O(G) = 1$. And as the lemma obviously holds for J_1, we assume that $G \not\cong J_1$.

By checking the character tables in the Atlas [$\mathbb{A}$], G has exactly one conjugacy class of 2-central involutions or $G \cong Fi_{23}$ and there are three conjugacy classes of 2-central involutions; also we observe that in any case the 2-central involutions are in $O^2(G)$. Therefore in a counterexample to the lemma there are involutions s and $t \in Z(S)$ with s and t conjugate in G. But then s and t are already conjugate in $N_G(S)$ and hence $N_G(S) > S$. Thus Lemma 2.21 implies that $G \cong J_2$, J_3, Suz, or HN. However, in each of these cases there exists $G \geq K \geq S$ with $C_K(O_2(K)) \leq O_2(K)$ and $O_2(K)$ an extraspecial 2-group (K being, respectively, $2^{1+4}A_5$, $2^{1+4}A_5$, $2^{1+6}U_4(2)$, $2^{1+8}(A_5 \times A_5):2$), and this implies $\Omega_1(Z(S)) = \Omega_1(Z(O_2(K))) \cong E(2)$, which contradicts our assumption that there are at least two involutions in $\Omega_1(Z(S))$. Hence Lemma 2.22 holds.

LEMMA 2.23. *Let X be a finite group, p a prime, V a $GF(p)X$-module, and $S \in Syl_p(X)$. Suppose that $V = \langle U^X \rangle$ for some subspace U of $C_V(S)$. Then $V = [V, X] + C_V(X)$.*

Proof. See [CD, 2.5].

LEMMA 2.24. *Suppose that V is a $GF(2)X$-module. If H is a $GF(2)$-hyperplane of V, then $[[V,t]:[H,t]] \leq 2$ for all involutions t in X.*

Proof. If $[V,t] \leq H$, then H is t-invariant and we are done. Otherwise $V = H + [V,t]$ and $[V,t] = [H,t]$, so Lemma 2.24 is true.

We now state some results about the rank-1 Lie-type groups over $GF(2^n)$. We begin with some details of the structure of these groups and then continue with some $GF(2)$-module results. Thus until notified further we assume that $G \in \mathcal{L}ie_1$(even), $S \in Syl_2(G)$, K is a complement to S in $N_G(S)$, and V is is a nontrivial $GF(2)G$-module.

Recall that a $GF(2)L_2(2^n)$-module V is called a natural module if $C_V(G) = 0$, $\dim_{GF(2)} V = 2^{2n}$, and $[V,S,S] = 0$.

LEMMA 2.25. *The following are true*:

(i) *G operates doubly transitive on $\{N_G(T) | T \in Syl_2(G)\}$ and $N_G(S)$ is a maximal subgroup of G*;

(ii) *if $T \in Syl_2(G)$ and $S \neq T$, then $S \cap T = 1$*;

(iii) *if x is an involution of G and $x \notin S$, then $G = \langle S, x \rangle$*;

(iv) *K is cyclic, $Z(G) \leq K$, and K normalizes exactly two Sylow 2-subgroup of G.*

Proof. See [DS, 5.1].

LEMMA 2.26. *Suppose that $G \cong L_2(2^n)$. Then*

(i) *S is elementary abelian, $|S| = 2^n$, $|K| = 2^n - 1$, and K operates transitively on $S \setminus \{1\}$*;

(ii) *if $A \leq G$ is a fours group and $t \in A^{\#}$, then there exists $g \in G$ such that $G = \langle t, A^g \rangle$.*

Proof. See [DS, 5.2].

LEMMA 2.27. *Suppose that $G \cong U_3(2^n)$ or $SU_3(2^n)$. Then*

(i) *$Z(S) = \Phi(S) = \Omega_1(S)$ is elementary abelian of order 2^n and $|S/Z(S)| = 2^{2n}$*;

(ii) *all the involutions of S are conjugate*;

(iii) *if $S \neq T \in Syl_2(G)$, then $\langle Z(S), Z(T) \rangle \cong SL_2(2^n)$*;

(iv) *if U is a fours group in G and $x \in U^{\#}$, then there exists $g, h \in G$, such that $G = \langle x, U^g, U^h \rangle$*;

(vi) $|K/Z(G)| = (2^{2n} - 1)/(2^n + 1, 3)$ *and, if* $n \geq 2$, *then* $K/Z(G)$ *acts irreducibly on* $S/Z(S)$ *and* $Z(S)$.

Proof. See [DS, 5.4 and the table before 5.2].

LEMMA 2.28. *Suppose that* $G \cong Sz(2^n)$, $n > 1$. *Then*

(i) n *is odd and* 3 *does not divide the order of* G;

(ii) S *is special of order* 2^{2n} *with* $|Z(S)| = 2^n$, *all the involutions of* S *are contained in* $Z(S)$ *and all the involutions of* G *are conjugate*;

(iii) *if* $S \neq T \in Syl_2(G)$, *then* $\langle Z(S), Z(T) \rangle = G$;

(iv) $|K| = 2^n - 1$ *and* $C_S(k) = 1$ *for all* $k \in K^{\#}$;

(v) *if* U *is a fours group in* G *and* $x \in U$, *then there exists* $g \in G$ *such that* $G = \langle x, U^g \rangle$.

Proof. See [DS, 5.3 and the table before 5.2].

Recall that $Sz(2) \cong Frob(20) \cong C_5 : C_4$, so the Sylow 2-subgroup S is cyclic of order 4. But $\Omega_1(Z(S))$ has order 2 and contains the involution of S.

LEMMA 2.29. *Suppose that* $A \leq S$ *and* $[V, A, A] = 0$. *Then* $A \leq \Omega_1(Z(S))$ *and* $|A| \leq |\Omega_1(Z(S))| \leq q = 2^n$.

Proof. Since A acts quadratically, it is elementary abelian. Thus, $A \leq \Omega_1(S) = \Omega_1(Z(S))$ by Lemmas 2.26(i), 2.27(i), and 2.28(ii).

LEMMA 2.30. *Suppose that* V *is a nontrivial irreducible* $GF(2)G$-*module*.

(i) *If* $G \cong L_2(2^n)$, *then* $|V| \geq q^2 = 2^{2n}$.

(ii) *If* $G \cong Sz(2^n)$, *then* $|V| \geq q^4 = 2^{4n}$.

(iii) *If* $G \cong U_3(2^n)$ *or* $SU_3(2^n)$, *then* $|V| \geq q^6 = 2^{6n}$.

Proof. See [DS, 5.7].

LEMMA 2.31. *Suppose that* $n \geq 2$, V *is a nontrivial irreducible* $GF(2)G$-*module, and* F *is a fours subgroup of* G *with* $t \in F^{\#}$. *Then the following statements hold*:

(i) *If* $G \cong L_2(2^n)$, *then* $[V : C_V(t)] \geq 2^n$.

(ii) *If* $G \cong Sz(2^n)$, $SU_3(2^n)$, *or* $U_3(2^n)$, *then* $[V : C_V(F)] \geq 2^{2n}$.

(iii) *If* $G \cong Sz(2^n)$, $SU_3(2^n)$, *or* $U_3(2^n)$ *and* $[V, F, F] = 0$, *then* $|V : C_V(t)| \geq 2^{2n}$.

(iv) *If* $G \cong Sz(2^n)$, $SU_3(2^n)$, *or* $U_3(2^n)$ *and* $n \geq 2$, *then* $[V : C_V(t)] > 2^n$.

Proof. See [CD, 2.9] for (i) and [DS, 5.10 b)] for (iii). That leaves (ii) and (iv). Suppose that $G \cong Sz(2^n)$, $SU_3(2^n)$, or $U_3(2^n)$, and $|V : C_V(F)| < 2^{2n}$. Then, by Lemmas 2.27(iv) and 2.28(v), if $G \cong Sz(2^n)$, $|V| < 2^{4n}$, and, if $G \cong SU_3(2^n)$ or $U_3(2^n)$, then $|V| < 2^{6n}$. This, of course, is against Lemma 2.30. Therefore (ii) is true. Because, by Lemma 2.27(iv) and Lemma 2.28(v), $SU_3(2^n)$ and $U_3(2^n)$ are generated by five involutions and $Sz(2^n)$ is generated by three involutions, (iv) follows from Lemma 2.30.

LEMMA 2.32. *Suppose that $A \leq S$, $C_V(O^2(G)) = 1$, $[V, A, A] = 0$, and $|A|^2 > [V : C_V(A)]$. Then $G \cong L_2(2^n)$ and V is a natural $GF(2)L_2(2^n)$-module. In particular, if V is FF-module with offending subgroup A, then $G \cong L_2(2^n)$ and $A = S$.*

Proof. See [DS, 5.12 b)].

LEMMA 2.33. *Suppose that $G \cong L_2(2^n)$, $V = \langle C_V(S)^G \rangle$, and $C_V(G) = 0$. If $[V, S, S] = 0$, then V is a direct sum of natural $GF(2)G$-modules.*

Proof. This is given in [CD, 2.11].

LEMMA 2.34. *Assume that x is an involution in G. Then $C_V(S) \cap C_V(x) \neq 0$ if and only if $x \in Z(S)$.*

Proof. Put $X = C_V(S) \cap C_V(x)$ and assume that $X \neq 0$. Then X is centralized by $\langle S, x \rangle$. Since V is irreducible we conclude from Lemmas 2.25(iii), 2.26, 2.27(i), and 2.28(ii) that $x \in Z(S)$. Conversely, if $x \in S$, then $x \in \Omega_1(Z(S))$, so $C_V(x)$ is S-invariant and therefore, $X \neq 0$.

LEMMA 2.35. *Suppose that $G \cong L_2(2^n)$, $1 \neq t \in S \in Syl_2(G)$, and V is a natural $GF(2)G$-module. Then $C_V(t) = C_V(S) = [V, S] = [V, t]$.*

Proof. This is immediate as V is a two-dimensional module over $GF(2^n)$.

LEMMA 2.36. *Suppose that $G \cong SU_3(2)$ or $U_3(2)$, $S \in Syl_2(G)$, and V is a faithful irreducible $GF(2)G$-module. Then*

(i) $\dim_{GF(2)} V \geq 6$;

(ii) *if $G \cong U_3(2)$, then V is the eight-dimensional irreducible permutation module obtained from its permutation action on the Sylow 2-subgroups of G;*

(iii) *suppose that $[V : C_V(t)] = 2^2$ for some involution $t \in G$; then $\dim_{GF(2)} V = 6$ and $G \cong SU_3(2)$;*

(iv) *suppose that $\dim_{GF(2)} V = 6$; then V is a three-dimensional $GF(4)$-module, $\dim_{GF(4)}[V, S] = 2.\dim_{GF(4)}[V, S, S] = 2$, and $[V, S, Z(S)] = 0$;*

(v) *If $\dim_{GF(2)} V > 6$, then $\dim_{GF(2)} V \geq 8$.*

Proof. From [Hu, p. 245] we recall that $|U_3(2)| = 2^3 3^2$, $|SU_3(2)| = 2^3 3^3$, and $U_3(2), SU_3(2)$ are both 3-closed with $S \cong Q_8$. For $R \in Syl_3(G)$, $R/Z(G) \cong E(3^2)$ and S acts transitively (by conjugation) on $(R/Z(G))^{\#}$. Part (i) follows from Lemma 2.30(iii). Employing a result of Berman [HB, 3.11] shows that if $G \cong U_3(2)$, then G has exactly two irreducible $GF(2)G$-modules. Since the eight-dimensional permutation module obtained from the action of G on $Syl_2(G)$ is irreducible, we have (ii). Since the involutions of G invert all the elements of $(R/Z(G))^{\#}$ and $SU_3(2)$ is a nonsplit central extension of $U_3(2)$ by C_3, it follows that we can find three involutions x, y, z of G such that $|\langle x, y, z\rangle| = 2|R|$ and $\langle x, y, z\rangle \lhd G$. So $C_V(\langle x, y, z\rangle)$ is G-invariant, whence, using (i), $\dim V = 6$. Then, by (ii), $G \cong SU_3(2)$, so (iii) holds. Part (iv) holds because the action of $Z(G)$ on the six-dimensional $GF(2)$-module, V, imposes a $GF(4)$ structure. If (v) is false, then $\dim_{GF(2)} V = 7$, and, by (ii), $Z(G) \cong C_3$ with $Z(G)$ acting fixed-point freely in V. But then 3 divides $2^7 - 1$, a contradiction.

LEMMA 2.37. *Suppose that $G \cong Sz(2) \cong Frob(20)$ and V is a faithful, irreducible $GF(2)G$-module. Then $\dim_{GF(2)} V = 4$, $[V : C_V(t)] = 2^2$ for all involutions $t \in G$ and every nontrivial vector in V is centralized by exactly one involution in G.*

Proof. It follows from [HB, 3.11] that G possesses, up to isomorphism, exactly two irreducible $GF(2)G$-modules. Therefore, as the action of G on $Syl_2(G)$ provides an irreducible module of dimension 4 on which the elements of order 5 operate fixed-point freely and because two involutions generate a dihedral group of order 10, we have the result.

LEMMA 2.38. *Suppose that $G \cong D_{2p}$, where p is an odd prime, t is an involution in G, and V is a nontrivial faithful irreducible $GF(2)G$-module. Then $[V : C_V(t)] \geq |V|^{1/2}$. Moreover, if V is an FF module for G, then $|V| = 2^2$, and $G \cong S_3 \cong SL_2(2)$.*

Proof. Because $G = \langle t, t^g\rangle$ for any $g \in G \backslash \langle t\rangle$, the result is immediate.

3. PRELIMINARY RESULTS AND THE PUSHING-UP CASE

In this section we start the proof of Theorem A. Thus from now on we assume that the following hypothesis holds:

HYPOTHESIS 3.1. *G is the free amalgamated product of two proper finite subgroups P_1 and P_2 over $P_1 \cap P_2$ (here groups are identified with their* *images in the free amalgamated product) which satisfy the following conditions*:

(1) $O^{2'}(P_1)/O_2(O^{2'}(P_1)) \in \mathsf{L}$ *and* $O^{2'}(P_2)/O_2(O^{2'}(P_2)) \in \mathsf{S}$;

(2) $B := P_1 \cap P_2$ *contains a Sylow* 2-*subgroup of both* P_1 *and* P_2;

(3) $P_i = O^{2'}(P_i)B$ *for* $i = 1, 2$;

(4) $C_{P_i}(O_2(P_i)) \leq O_2(P_i)$ *for* $i = 1, 2$;

(5) *no nontrivial normal subgroup of* G *is contained in* B.

As we remarked in the Introduction, any group satisfying the hypotheses of Theorem A is a homomorphic image of the group satisfying Hypothesis 3.1. Therefore, to prove our theorem it suffices to classify the amalgams fulfilling Hypothesis 3.1.

In the situation described by Hypothesis 3.1 we can construct a graph Γ, the coset graph, which has vertices $V(\Gamma) = \{P_i g \mid g \in G,\ i = 1, 2\}$ with edge set $E(\Gamma) = \{(P_i g, P_j h) \mid P_i g \cap P_j h \neq \emptyset,\ i \neq j\}$. For $\delta, \lambda \in V(\Gamma)$ we shall say that δ, λ are adjacent if, and only if, $(\delta, \lambda) \in E(\Gamma)$. Clearly G acts on the graph Γ by right multiplication. Since G is the free amalgamated product $P_1 *_{P_1 \cap P_2} P_2$ and P_1 and P_2 are finite groups, Γ is a locally finite tree (see Serre [Se]). We draw the following result from [DS]:

PROPOSITION 3.2. *The following hold*:

(i) *G acts faithfully on* Γ;

(ii) *if* $\alpha \in V(\Gamma)$, *then* $G_\alpha := stab_G(\alpha)$ *is conjugate in* G *to one of* P_1 *or* P_2;

(iii) *If* $(\alpha, \beta) \in E(\Gamma)$, *then* $G_{\alpha\beta} := stab_G((\alpha, \beta))$ *is conjugate in* G *to* $P_1 \cap P_2$.

We next introduce the notation that will be maintained throughout the rest of this paper and [PR3]. Suppose that $\delta \in V(\Gamma)$ and $(\delta, \lambda) \in E(\Gamma)$. Then

$\Delta(\delta) = \{\gamma \in V(\Gamma) \mid (\gamma, \delta) \in E(\Gamma)\}$;

$O(\mathsf{L}) = \{\gamma \in V(\Gamma) \mid G_\gamma$ is G-conjugate to $P_1\}$;

$O(\mathsf{S}) = \{\gamma \in V(\Gamma) \mid G_\gamma$ is G-conjugate to $P_2\}$;

$L_\delta = O^{2'}(G_\delta)$;

$Q_\delta = O_2(L_\delta)$;

$K_\delta = \mathrm{core}_{L_\delta}(Q_\lambda)$;

$S_{\delta\lambda}$ is a Sylow 2-subgroup of $G_{\delta\lambda}$;

$Z_\delta = \langle \Omega_1(Z(S_{\delta\lambda}))^{L_\delta} \rangle$;
$V_\delta = \langle Z_\lambda^{L_\delta} \rangle$;
$U_\delta = \langle V_\lambda^{L_\delta} \rangle$;
$W_\delta = \langle U_\lambda^{L_\delta} \rangle$;
$D_\delta = C_{L_\delta}(V_\delta/[V_\delta, Q_\delta])$;
$T_\delta = O_{2,2'}(L_\delta)$.

Further, for $\delta \in O(\mathsf{S})$ we let $F_\delta \geq Q_\delta$ be such that $F_\delta/Q_\delta = F^*(L_\delta/Q_\delta)$.

Finally, if $\delta \in O(\mathsf{L})$ and $L_\delta/Q_\delta \notin \{S_5\} \cup \mathsf{D}$, then q_δ is the order of the field of definition of L_δ/Q_δ; if $L_\delta/Q_\delta \cong S_5$, then $q_\delta = 4$; if $L_\delta/Q_\delta \in \mathsf{D}$, then $q_\delta = 2$.

LEMMA 3.3. *Suppose that $\delta \in V(\Gamma)$. Then L_δ is transitive on $\Delta(\delta)$. Furthermore, if $\delta \in O(\mathsf{S})$, then F_δ is transitive on $\Delta(\delta)$.*

Proof. Assume that $\delta \in V(\Gamma)$. Then by Proposition 3.2(ii) we may suppose that $\delta \in \{P_1, P_2\}$. Since $\Delta(\delta)$ consists of cosets of B in P_1 $(i \in \{1,2\})$, G_δ is transitive on $\Delta(\delta)$, whence, by Hypothesis 3.1(3), L_δ is also transitive on $\Delta(\delta)$. Moreover, if $\delta \in O(\mathsf{S})$, then from the Atlas [$\mathbb{A}$] $[L_\delta : F_\delta] \leq 2$ and so $L_\delta = S_{\gamma\delta} F_\delta$. Therefore F_δ is transitive on $\Delta(\delta)$.

The following result will be important in our later arguments.

LEMMA 3.4. *Suppose that $(\delta, \lambda) \in E(\Gamma)$ and $N \leq G_{\delta\lambda}$. If $N_{G_\delta}(N)$ is transitive on $\Delta(\delta)$ and $N_{G_\lambda}(N)$ is transitive on $\Delta(\lambda)$, then $N = 1$.*

Proof. See [DS, 2.6].

LEMMA 3.5. *Suppose that $\delta \in V(\Gamma)$. Then*

(i) $Z_\delta \leq \Omega_1(Z(Q_\delta))$;

(ii) *if $Z_\delta \neq \Omega_1(Z(L_\delta))$, then $C_{L_\delta}(Z_\delta) \leq T_\delta$ and $Q_\delta \in Syl_2(C_{L_\delta}(Z_\delta))$;*

(iii) *if X is a 2-subgroup of L_δ with $X \nleq Q_\delta$ and $[X, Z_\delta] = 1$, then $Z_\delta = \Omega_1(Z(L_\delta))$.*

Proof. From Hypothesis 3.1(4), $\Omega_1(Z(S_{\delta\lambda})) \leq Q_\delta$ for any $\lambda \in \Delta(\delta)$ and hence $Z_\delta \leq \Omega_1(Z(Q_\delta))$.

(ii) Since $C_{L_\delta}(Z_\delta)$ is a proper normal subgroup of L_δ (and $L_\delta = F_\delta S_{\delta\lambda}$ when $\delta \in O(\mathsf{S})$ for any $\lambda \in \Delta(\delta)$), we deduce from the structure of L_δ/Q_δ given in Hypothesis 3.1(1) that $C_{L_\delta}(Z_\delta) \leq T_\delta$. By part (i), $Q_\delta \leq C_{L_\delta}(Z_\delta)$, so $Q_\delta \in Syl_2(C_{L_\delta}(Q_\delta))$. Thus (ii) holds and (iii) follows directly from (ii).

We define $d(\ ,\ )$ to be the standard distance function on Γ. Then the critical distance b is defined by:

$$b = \min_{\tau,\rho \in V(\Gamma)} \left\{ d(\tau, \rho) \mid Z_\tau \nleq Q_\rho \right\}.$$

If $(\tau, \rho) \in V(\Gamma) \times V(\Gamma)$ is a pair of vertices with $d(\tau, \rho) = b$ and $Z_\tau \nleq Q_\rho$, then (τ, ρ) is called a critical pair. The set of all critical pairs will be denoted by $\mathcal{C}$. Observe that, since $Z_\delta \neq 1$ for $\delta \in V(\Gamma)$, Proposition 3.2(i) implies that b is finite. Further, by Lemma 3.5(i), $b \geq 1$. Whenever we fix $(\alpha, \alpha') \in \mathcal{C}$, we will denote the vertices of the unique path joining α and α' as follows:

$$\alpha, \beta, \alpha + 2, \ldots, \alpha' - 1, \alpha'.$$

The proof of the following proposition is straightforward and is left to the reader.

PROPOSITION 3.6. *The following hold*:

(i) *if* $(\alpha, \alpha') \in \mathcal{C}$, *then* $C_{L_\alpha}(Z_\alpha) \leq T_\alpha$;

(ii) *if* $(\alpha, \alpha') \in \mathcal{C}$ *and* $[Z_\alpha, Z_{\alpha'}] \neq 1$, *then* $(\alpha', \alpha) \in \mathcal{C}$;

(iii) *if* $(\alpha, \alpha') \in \mathcal{C}$ *and* $[Z_\alpha, Z_{\alpha'}] = 1$, *then* $Z_{\alpha'} = \Omega_1(Z(L_{\alpha'}))$; *in particular*, b *is odd and* $C_{Z_\alpha}(L_\alpha) = 1$.

LEMMA 3.7. *Let* $(\delta, \lambda) \in E(\Gamma)$ *and suppose that* Z *is a nontrivial normal subgroup of* L_δ *contained in* Z_δ. *Put* $V = \langle Z^{L_\lambda} \rangle$, $U = \langle V^{L_\delta} \rangle$, *and* $W = \langle U^{L_\lambda} \rangle$. *If* $b \geq 3$ (*respectively*, $b \geq 5$, $b \geq 7$), *then* V (*respectively*, U, W) *is elementary abelian. Furthermore*,

(i) *if* $b \geq 2$, *then* $V \leq Q_\lambda$ *and* $\lambda(L_\lambda, V/[V, Q_\lambda]) \geq 1$;

(ii) *if* $b \geq 3$, *then* $U \leq Q_\delta$ *and* $\eta(L_\delta, U/[U, Q_\delta]Z) \geq 1$;

(iii) *if* $b \geq 4$, *then* $W \leq Q_\lambda$ *and* $\eta(L_\lambda, W/[W, Q_\lambda]V) \geq 1$.

Proof. The first part follows easily from the definition of b, V, U, and W. As the same type of argument establishes each of parts (i), (ii), and (iii), we just prove (iii). Again it is immediate from its definition that $W \leq Q_\lambda$. Now suppose that $\eta(L_\lambda, W/[W, Q_\lambda]V) = 0$. Then $W = [W, Q_\lambda]VU = [W, Q_\lambda]U$ and hence

$$[W, Q_\lambda] = [[W, Q_\lambda]U, Q_\lambda] = [[W, Q_\lambda], Q_\lambda][U, Q_\lambda].$$

Therefore,

$$W = [W, Q_\lambda]U = [[W, Q_\lambda], Q_\lambda][U, Q_\lambda]U = [[W, Q_\lambda], Q_\lambda]U.$$

By repeatedly commutating with Q_λ it follows that $W = U$, whence, by Lemma 3.3 and Lemma 3.4, $W = U = 1$. But this is against $Z \neq 1$. Hence $\eta(L_\lambda, Q/[W, Q_\lambda]V) \geq 1$.

Our next lemma is a variant of Lemma 3.7(ii) in the case when $\lambda \in O(\mathcal{S})$.

LEMMA 3.8. *Let* $(\delta, \lambda) \in E(\Gamma)$ *and let* Z, V, *and* U *be as in Lemma* 3.7. *If* $\lambda \in O(\mathsf{S})$, $S_{\delta\lambda} = Q_\delta Q_\lambda$, *and* $b \geq 3$, *then either* $\eta(L_\delta, U/Z) \geq 2$, *or* $C_{L_\lambda}(V) \cap Q_\delta \leq K_\delta$.

Proof. By Lemma 3.7, $\eta(L_\lambda, V/[V, Q_\lambda]) \geq 1$ and $\eta(L_\delta, U/Z[U, Q_\delta]) \geq 1$. Assume that $\eta(L_\delta, Z[U, Q_\delta]/Z) = 0$. Then $[U, Q_\delta]Z = [V, Q_\delta]Z$. Put $X = C_{L_\lambda}(V) \cap Q_\delta$, and let $\tau \in \Delta(\delta)$. Clearly $X \leq L_\tau$ and $\tau \in O(\mathsf{S})$. By Lemma 3.3, we may choose $g \in L_\delta$ so as $\lambda.g = \tau$. Setting $V_1 = V^g$ we observe that X centralizes $[V_1, Q_\delta]$ since $[V_1, Q_\delta] \leq [U, Q_\delta]Z = [V, Q_\delta]Z \leq V$. Put $\overline{V}_1 = V_1/[V_1, Q_\tau]$. From $S_{\tau\delta} = Q_\tau Q_\delta$ we get $[\overline{V}_1, S_{\tau\delta}] = [\overline{V}_1, Q_\delta]$ is centralized by X. Now, $X \lhd Q_\delta$ implies that $XQ_\tau/Q_\tau \lhd S_{\tau\delta}/Q_\tau$. Since $\eta(L_\tau, \overline{V}_1) \geq 1$, Lemma 2.16 forces $X \leq Q_\tau$ and hence, as τ was an arbitrary element of $\Delta(\delta)$, $X \leq K_\delta$. This proves the lemma.

The next two results are drawn from Stellmacher's article [St2]; first we establish his notation.

Suppose H is a group, S is a 2-subgroup of H and V is a $GF(2)H$-module. Then

$$Q(S,V) = \{A \leq S \mid A \text{ acts quadratically on } V\}.$$

$$q(S,V) = \begin{cases} \min\left\{\log_{|A/C_A(V)|}\left(|V/C_V(A)|\right) \mid A \in Q(S,V)\right\}, \\ \qquad\qquad \text{when } Q(S,V) \neq \varnothing, \\ 0, \qquad\quad \text{when } Q(S,V) = \varnothing. \end{cases}$$

Now suppose that $(\alpha, \alpha') \in \mathsf{C}$ and let $1 = V_0 \leq V_1 \leq \cdots \leq V_n = V_\beta$ be a chief series of $GF(2)L_\beta$-submodules. Set

$$c = \eta(L_\beta, V_\beta);$$

$$q = q(Q_\beta, Z_\alpha);$$

$$r = \min\left\{q(S_{\alpha\beta}, V_i/V_{i-1}) \mid q(S_{\alpha\beta}, V_i/V_{i-1}) \neq 0 \text{ and } 1 \leq i \leq n\right\}.$$

LEMMA 3.9. *Suppose that* $(\alpha, \alpha') \in \mathsf{C}$, $b \geq 3$, $[Z_\alpha, Z_{\alpha'}] = 1$, *and* $V_{\alpha'} \not\leq Q_\beta$. *If* $q - 1 \geq 0$ *and* $rc - 1 \geq 0$, *then* $(q-1)(rc-1) \leq 1$.

Proof. Since $b \geq 3$, $V_{\alpha'}$ acts quadratically on V_β and V_β acts quadratically on $V_{\alpha'}$. Also, because the situation is symmetric we may assume that $|V_\beta/C_{V_\beta}(V_{\alpha'})| \leq |V_{\alpha'}/C_{V_{\alpha'}}(V_\beta)|$. Thus [St2, 3.1 c)] is applicable and yields the result.

LEMMA 3.10 [St2]. *Suppose that* $(\alpha, \beta) \in E(\Gamma)$, $b \geq 2$, *and* $\eta(L_\beta, V_\beta) = 1$. *Set* $Q = [O^2(L_\beta), Q_\beta]$ *and assume that* $Q \not\leq C_{L_\alpha}(Z_\alpha)$. *Then* Q *operates quadratically on* Z_α *and* $\|[Z_\alpha, Q]\| \leq |QC_{L_\alpha}(Z_\alpha)/C_{L_\alpha}(Z_\alpha)|$.

Proof. By Lemma 3.7(i), $\eta(L_\beta, V_\beta/[V_\beta, Q_\beta]) \geq 1$. Therefore, as $\eta(L_\beta, V_\beta) = 1$, $[V_\beta, Q_\beta, O^2(L_\beta)] = 1$. Because $Q \leq O^2(L_\beta)$, we thus have $[V_\beta, Q, Q] = 1$, and so, in particular, Q operates quadratically on Z_α. Further, as $[Q, O^2(L_\beta)] = Q$, the three subgroup lemma gives

$$\left[V_\beta, O^2(L_\beta), Q\right] = \left[V_\beta, \left[Q, O^2(L_\beta)\right]\right] = \left[V_\beta, Q\right]. \qquad (*)$$

Now set $V_0 = C_{V_\beta}(O^2(L_\beta))$, and select $v \in Z_\alpha \setminus V_0$. Then $[V_\beta, O^2(L_\beta)] \leq \langle v^{O^2(L_\beta)}\rangle V_0$ and $V_\beta = \langle v^{O^2(L_\beta)}\rangle V_0 Z_\alpha$. Therefore, using $(*)$ we have

$$\begin{aligned}[Z_\alpha, Q] \leq \left[V_\beta, Q\right] = \left[V_\beta, O^2(L_\beta), Q\right] &\leq \left[\langle v^{O^2(L_\beta)}\rangle V_0, Q\right] \\ &= \left[\langle v^{O^2(L_\beta)}\rangle, Q\right] = [v, Q]^{O^2(L_\beta)} = [v, Q] \leq [Z_\alpha, Q].\end{aligned}$$

Hence $[Z_\alpha, Q] = [v, Q]$. Since $[v, Q] \leq Z(Q)$ the map $x \mapsto [v, x]$ is a surjective group homomorphism from Q to $[v, Q]$ and consequently $[v, Q] \cong Q/C_Q(v)$. In particular, $\|[Z_\alpha, Q]\| = \|[v, Q]\| = [Q : C_Q(v)]$ from which we infer that $\|[Z_\alpha, Q]\| \leq |QC_{L_\alpha}(Z_\alpha)/C_{L_\alpha}(Z_\alpha)|$.

Roughly speaking Lemma 3.10 tells us that if V_β contains only one noncentral chief factor then Z_α is the $GF(2)$-dual of a failure to factorise module. Thus we get

LEMMA 3.11. *Suppose that* $(\alpha, \alpha') \in \mathcal{C}$, $b \geq 2$, $\eta(L_\beta, V_\beta) = 1$, *and* $C_{L_\alpha}(Z_\alpha) = 1$. *Set* $Q = [O^2(L_\beta), Q_\beta]$ *and assume that* $Q \nleq C_{L_\alpha}(Z_\alpha)$. *Then* $L_\alpha/Q_\alpha \cong L_2(q_\alpha)$ *or* S_5 *and* Z_α *is either a natural* L_α/Q_α*-module or an orthogonal* S_5*-module.*

Proof. This follows from Lemmas 3.10 and 2.3, Theorem 2.15(iii), and Lemmas 2.32 and 2.38 (and the fact that the dual of an FF L_α/Q_α-module is also an FF L_α/Q_α-module when $L_\alpha/Q_\alpha \in \mathcal{L}ie_1(\text{even})$).

The next results of this section are concerned with establishing Theorem 3.20. The proof relies on certain results related to pushing-up problems. Hence for the next five results we suppose that X is a finite group for which $C_X(O_2(X)) \leq O_2(X)$, $S \in Syl_2(X)$, $Z = \langle \Omega_1(Z(S))^X\rangle$, and, for $O_2(X) \leq T \lhd S$, $B(T) = C_T(\Omega_1(Z(J(T))))$ is the Baumann subgroup of T.

LEMMA 3.12. *Suppose that* $X/O_2(X) \cong L_2(2^n)$ *and that no nontrivial characteristic subgroup of* S *is normal in* X. *Then* $[O_2(X), O^2(X)] \leq Z$, *and* $Z/\Omega_1(Z(X))$ *is a natural* $L_2(2^n)$*-module.*

Proof. See [B2], [N], or [St1].

PROPOSITION 3.13. *Suppose that* $X/O_2(X) \cong L_2(2^n)$, $H \leq \mathrm{Aut}(S)$, *and that no nontrivial characteristic subgroup of* S *is normal in* X. *Set* $N =$

$N_H(O_2(X))$. *If* $y \in H$ *is such that* $[Z, Z^y] \neq 1$, *then* $N = N_H(O_2(X)^y)$ *and* $|\langle y \rangle N/N| = 2$; *in particular*, y *has even order*.

Proof. Suppose that $y \in H$ satisfies $[Z, Z^y] \neq 1$, and set $Q = O_2(X)$ and $R = Q^y \cap Q$. It follows from $[Z, Z^y] \neq 1$ that $Z^y \nleq Q$, and hence $y \notin N$. Without loss of generality we may assume that $|Z^y Q/Q| \geq |ZQ^y/Q^y|$. Then, noting that $C_S(Z^y) = Q^y$,

$$|Z^y Q/Q| \geq |ZQ^y/Q^y| = [Z : Z \cap Q^y] = [Z : C_Z(Z^y)]. \qquad (*)$$

Thus $Z^y Q/Q$ is an offending subgroup on Z and hence equality holds in $(*)$ by Lemma 2.32, with $Z^y Q = S$. From $[Z : Z \cap Q^y] = [S : Q] = [S : Q^y]$ we also get $S = ZQ^y$. Using Dedekind's modular law gives $Q = Q \cap ZQ^y = ZR$ and $Q^y = Q^y \cap Z^y Q = Z^y R$. Therefore,

$$(3.13.1) \qquad \Phi(Q) = \Phi(ZR) = \Phi(R) = \Phi(Z^y R) = \Phi(Q^y).$$

Set $\bar{S} = S/\Phi(R)$. Then $\bar{Q}$ and $\bar{Q}^y$ are elementary abelian subgroups of $\bar{S}$. From a lemma of Burnside's [Gor, Theorem 5.1.4] and Lemma 3.12, we have $\eta(X, \bar{Q}) = 1$ with the X-noncentral chief factor in $\bar{Q}$ being a natural $L_2(2^n)$-module. Note that $S = QQ^y$ gives $\bar{S} = \bar{Q}\overline{Q^y}$ and $[\bar{Q} : \bar{R}] = [Q : R] = [S : Q^y] = 2^n$. Using Lemma 2.35 we infer that $\bar{R} = C_{\bar{Q}}(\overline{Q^y}) = C_{\bar{Q}}(\bar{x})$ for all $\bar{x} \in \bar{S} \setminus \bar{Q}$. Consequently, by [LPR, Lemma 3.12], $\bar{Q}$ and $\bar{Q}^y$ are the only maximal elementary abelian subgroups of $\bar{S}$. Since N normalizes Q and hence normalizes $\Phi(Q) = \Phi(R)$, N acts upon $\bar{S}$. Because N normalizes $\bar{Q}$, it must then normalize $\bar{Q}^y$ and hence N normalizes Q^y. Thus we conclude that $N = N_H(O_2(X)^y)$.

Observe that, using (3.13.1), $\Phi(R)^y = \Phi(Q)^y = \Phi(Q^y) = \Phi(R)$ and, therefore, y also acts upon $\bar{S}$. Hence y^2 normalizes $\bar{Q}$, whence $y^2 \in N$ and this completes the proof of the proposition.

The next result is similar in spirit to that in [St1, Theorem C]; however, we do not require that $N_H(O_2(X))$ is normal in H.

COROLLARY 3.14. *Suppose that* $X/O_2(X) \cong L_2(2^n)$, *that no nontrivial characteristic subgroup of* S *is normal in* X, *and that* $H \leq \mathrm{Aut}(S)$ *with* $[H : N_H(O_2(X))]$ *odd. Then* $\langle Z^H \rangle$ *is a normal subgroup of* X *contained in* $O_2(X)$.

Proof. Suppose there exists $h \in H$ such that $Z^h \nleq O_2(X)$. Then, by Proposition 3.13, $h^2 \in N_H(O_2(X))$ and $\langle h \rangle N_H(O_2(X))$ is a subgroup with $[\langle h \rangle N_H(O_2(X)) : N_H(O_2(X))] = 2$. This is against our hypothesis that $[H : N_H(O_2(X))]$ is odd. Therefore, as $Z \leq \langle Z^H \rangle \leq O_2(X)$, Lemma 3.12 implies that $\langle Z^H \rangle$ is normal X.

The next two lemmas allow us to by-pass certain pushing-up configurations which involve S_5.

LEMMA 3.15. *Suppose that* $X/O_2(X) \cong L_2(2^n)$, *or* $X/O_2(X) \cong S_5$ *and* $Z/C_Z(O^2(X))$ *is a natural* $GF(2)S_5$*-module. If* $J(T) \nleq O_2(X)$, *then* $B(T) \in Syl_2(\langle B(T)^X\rangle)$ *and* $\langle B(T)^X\rangle/O_2(\langle B(T)^X\rangle) \cong L_2(2^m)$, *where* $m \in \{n, 2\}$.

Proof. Compare with [B1, 2.11.1.4] and [B2, (7)]. Set $Q = O_2(X)$, $X_0 = O^2(X)Q$, $S_0 = S \cap X_0$, and $R = \langle \Omega_1(Z(J(T)))^X\rangle$. Put $q = 2^n$ if $X/Q \cong L_2(2^n)$ and $q = 4$ if $X/Q \cong S_5$. Then, since $J(T) \nleq Q$, we can pick $A \in \mathcal{A}(T)$ such that $A \nleq Q$. Suppose that $X/Q \cong S_5$. Then, as $Z/C_Z(O^2(X))$ is a natural S_5-module by hypothesis, we infer, from Lemma 2.3, that $J(T)Q = AQ = S_0$. While if $X/O_2(X) \cong L_2(2^n)$, then Lemma 2.32 implies that $J(T)Q = AQ = S_0$. Now, because $A \in \mathcal{A}(T)$, Lemmas 2.32 and 2.3(vi)(a) imply that $|AQ/Q| = |Z/C_Z(A)| = |Z/(Z \cap A)| = q$. Hence

(3.15.1) (i) $(A \cap Q)Z \in \mathcal{A}(T) \cap \mathcal{A}(Q)$ and (ii) $[Z : A \cap Z] = |AQ/Q| = q$ with $\eta(X, Z) = 1$.

Moreover, for any $A^* \in \mathcal{A}(T)$ with $A^* \nleq Q$ we also have $(A^* \cap Q)Z \in \mathcal{A}(T)$ and consequently $A \cap Z \leq \Omega_1(Z(J(T)))$. Hence $Z \leq \langle \Omega_1(Z(J(T)))^X\rangle = R$. Notice that, by (3.15.1)(i), $J(Q) \leq J(T)$, so $\Omega_1(Z(J(T))) \leq \Omega_1(Z(J(Q)))$ and, in particular, $R \leq \Omega_1(Z(J(Q)))$. Therefore we have $R \leq (A \cap Q)Z$, and so, $Z \leq R$ and Dedekind's modular law then give $R = R \cap Z(A \cap Q) = Z(A \cap R)$. Thus (3.15.2)(ii) implies that

$$|R/(A \cap R)| = |Z(A \cap R)/(A \cap R)| = |Z/(Z \cap A)| = q.$$

It follows that $\eta(X, R) = 1$ and hence $R = Z\Omega_1(Z(J(T)))$. Therefore (as $A \cap Z \leq \Omega_1(Z(J(T)))$),

$$\begin{aligned} R \cap A &= Z\Omega_1(Z(J(T))) \cap A = (Z \cap A)\Omega_1(Z(J(T))) \\ &= \Omega_1(Z(J(T))). \end{aligned}$$

Now choose $x \in X$ such that $\langle A, A^x\rangle Q \geq X_0$. Then $R_0 = R \cap A \cap A^x = \Omega_1(Z(J(T))) \cap \Omega_1(Z(J(T^x)))$ is normalized by X_0 and $[R : R_0] = q^2$. Since $\eta(X_0, R/R_0) = 1$, $R = ZR_0$ which then gives

$$\Omega_1(Z(J(T))) = R \cap A = ZR_0 \cap A = (Z \cap A)R_0,$$

and so $\Omega_1(Z(J(T))) = \Omega_1(Z(S_0))R_0$. Set $C = C_{X_0}(R_0)$ and $S_1 = S_0 \cap C \in Syl_2(C)$. Then $C \lhd X_0$ and $S_1 \in Syl_2(C)$. Clearly, S_1 centralizes $\Omega_1(Z(J(T)))$ and so $S_1 \leq B(T)$. Equally clearly $B(T) \leq C$ and thus we conclude that $B(T) = S_1 \in Syl_2(C)$ with (as $B(T) \lhd S$) $C = \langle B(T)_0^X\rangle = \langle B(T)^X\rangle$. Since $C/O_2(C) \cong X_0/Q$, the lemma is proven.

LEMMA 3.16. *Suppose that* $X/O_2(X) \cong S_5$, $Z/C_Z(O^2(X))$ *is an orthogonal module, and* $J(T) \not\leq O_2(X)$. *Then there exists a subgroup* R *of* X *such that* $B(T) \in Syl_2(R)$, $R/O_2(R) \cong S_3 \cong L_2(2)$, *and* $\langle R, S\rangle = X$.

Proof. Put $\overline{X} = X/O_2(X)$. Then, since $J(T) \not\leq O_2(X)$ and $Z/C_Z(O^2(X))$ is an orthogonal module, Lemma 2.3(iii) and (v)(a) and the normality of T in S imply that $\overline{J(T)}$ is a fours group which is not contained in $O^2(\overline{X})$. Without loss of generality, we assume that $\overline{B(T)} = \overline{J(T)} = \langle(1,2),(1,2)(3,4)\rangle$. Choose $R_1 \geq O_2(X)$ such that $\overline{R}_1 = C_{\overline{X}}((1,2)) \geq \overline{B(T)}$. Then $R_1/O_2(R_1) \cong L_2(2)$, $\langle R_1, S\rangle = X$, and, for $S_1 = S \cap R_1 \in Syl_2(R_1)$, $J(S_1) = J(T) \not\leq O_2(R_1)$. The result now follows directly from Lemma 3.15 with $R = \langle B(T)^{R_1}\rangle$.

We now revert to our standard notation.

LEMMA 3.17. *Suppose that* $(\delta, \lambda) \in E(\Gamma)$ *and* $Q_\delta \cap Q_\lambda \lhd L_\delta$. *Then*

(i) $\eta(L_\delta, Q_\delta/(Q_\delta \cap Q_\lambda)) = 0$;

(ii) $Z_\delta \leq V_\delta = \langle Z_{\lambda\delta}^{L}\rangle \leq Q_\delta \cap Q_\lambda$; *in particular,* $b \geq 2$.

Proof. First observe that $Q_\lambda \not\leq Q_\delta$, for otherwise $Q_\lambda \lhd \langle L_\delta, L_\delta\rangle$, which then contradicts Lemma 3.4. Thus, since $[Q_\delta, \langle \delta_\lambda^{L_\delta}\rangle] \leq Q_\delta \cap Q_\lambda$, we infer that $\eta(L_\delta, Q_\delta/(Q_\delta \cap Q_\lambda)) = 0$. We next prove part (ii). Evidently, $[Q_\delta \cap Q_\lambda, Z_\lambda] = 1$, and so, if $Z_\lambda \not\leq Q_\delta$, then $\eta(L_\delta, Q_\delta \cap Q_\lambda) = 0$. This together with (i) then implies the untenable $\eta(L_\delta, Q_\delta) = 0$. Hence $Z_\delta \leq V_\delta = \langle Z_\lambda^{L_\delta}\rangle \leq Q_\delta \cap Q_\lambda$, as stated.

LEMMA 3.18. *Suppose that* $(\delta, \lambda) \in E(\Gamma)$ *and* $Q_\delta \cap Q_\lambda \lhd L_\delta$. *Set* $Z = \Omega_1(Z(Q_\delta \cap Q_\lambda))$. *Then*

(i) $L_\delta/Q_\delta \cong L_2(q_\delta)$ *or* S_5;

(ii) $[S_{\delta\lambda} : J(Q_\lambda)Q_\delta] = 1$ (*respectively,* 2) *if* $L_\delta/Q_\delta \cong L_2(q_\delta)$ (*respectively,* $L_\delta/Q_\delta \cong S_5$) *and so, in particular,* $[S_{\delta\lambda}, Q_\delta Q_\lambda] \leq 2$;

(iii) $\eta(L_\delta, Z) = 1$ *with* $Z/C_Z(O^2(L_\delta))$ *being either a natural* L_δ/Q_δ-*module or an orthogonal* S_5-*module*.

Proof. First we note that $J(Q_\lambda) \leq Q_\delta$ would imply that $J(Q_\lambda) = J(Q_\delta \cap Q_\lambda)$, which is a contradiction to Lemma 3.4. Therefore, there exists $A \in \mathcal{A}(Q_\lambda)$ such that $A \not\leq Q_\delta$. Since $(A \cap Q_\delta)\Omega_1(Z(Q_\delta \cap Q_\lambda))$ is elementary abelian, $|(A \cap Q_\delta)\Omega_1(Z(Q_\delta \cap Q_\lambda))| \leq |A|$. Hence AQ_δ/Q_δ acts as an offending subgroup on any noncentral chief factor within $\Omega_1(Z(Q_\delta \cap Q_\lambda))$. Now Lemma 3.17(ii) (as $V_\delta \leq \Omega_1(Z(Q_\delta \cap Q_\lambda))$) implies that $\eta(L_\delta, \Omega_1(Z(Q_\delta \cap Q_\lambda))) \neq 0$, so Lemmas 2.3, 2.32, and 2.38 imply that $L_\delta/Q_\delta \cong L_2(q_\delta)$, or S_5 and also yield part (iii) (recall that the orthogonal S_5-module is projective). If $L_\delta/Q_\delta \cong L_2(q_\delta)$, then $AQ_\delta = S_{\delta\lambda}$, by Lemma 2.32, and so $J(Q_\lambda)Q_\delta = S_{\delta\lambda}$. Now assume that $L_\delta/Q_\delta \cong S_5$. Because neither the natural nor the orthogonal S_5-modules possess 2-central transvections $|J(Q_\lambda)Q_\delta/Q_\delta| \neq 2$, so using Lemma 2.3 again, we see that $|J(Q_\lambda)Q_\delta/Q_\delta| = 2^2$. This completes the verification of the lemma.

We now set $L = \langle Q_\lambda^{L_\delta} \rangle$.

LEMMA 3.19. *Suppose that* $(\delta, \lambda) \in E(\Gamma)$, $Q_\delta \cap Q_\lambda \lhd L_\delta$, *and* $Z_\lambda \nleq Z(L_\lambda)$. *Then* $O_2(L) = Q_\delta \cap Q_\lambda$.

Proof. Together Lemmas 3.17(ii) and 3.18(iii) give $\eta(L_\delta, V_\delta) = 1$, and thus $[V_\delta, Q_\delta] = [Z_\lambda, Q_\delta]$ is centralized by L. In particular, $[Z_\lambda, Q_\delta]$ is centralized by $O_2(L)$. By Lemma 3.18(i), $\lambda \in O(\mathcal{S})$. Thus, since $\eta(L_\lambda, Z_\lambda) \neq 0$, by hypothesis, combining Proposition 2.16 and Lemma 3.18(ii) we get that $O_2(L) \leq Q_\lambda$. Therefore, $Q_\delta \cap Q_\lambda = O_2(L)$, as required.

THEOREM 3.20. *Suppose that* $(\delta, \lambda) \in E(\Gamma)$ *and* $Q_\delta \cap Q_\lambda$ *is normal in* L_δ. *Then* $Z_\lambda \leq Z(L_\lambda)$.

Proof. Assume by way of contradiction that $Z_\lambda \nleq Z(L_\lambda)$. Then, by Lemma 3.19, $O_2(L) = Q_\delta \cap Q_\lambda$. Put $S = S_{\delta\lambda} \cap L$. Then $O_2(L) \leq Q_\lambda \lhd S$. If $J(Q_\lambda) \leq O_2(L)$, then $J(Q_\lambda) = J(O_2(L))$ is normal in $\langle L_\lambda, L \rangle = \langle L_\lambda, L_\delta \rangle$, which is against Lemma 3.4. Therefore $J(Q_\lambda) \nleq O_2(L)$. Also observe that, if C is a normal subgroup of L which is characteristic in $B(Q_\lambda)$, then, as $B(Q_\lambda)$ is characteristic in Q_λ, C is normal in L_λ, in which case Lemma 3.4 implies that $C = 1$.

We consider the two cases which arise in Lemma 3.18(iii) separately. Set $Z = \Omega_1(Z(Q_\delta \cap Q_\lambda))$ and $E = \langle B(Q_\lambda)^L \rangle$. Assume first that $Z/C_Z(O^2(L))$ is a natural $L/O_2(L)$-module. Then, by Lemma 3.15, $B(Q_\lambda) \in Syl_2(E)$, $E/O_2(E) \cong L_2(2^n)$, and $EO_2(L) \geq O^2(L)$. Moreover, we have $O_2(E) = O_2(L) \cap B(Q_\lambda)$ and, since $Z \leq B(Q_\lambda)$, $C_E(O_2(E)) \leq O_2(E)$; therefore, we may apply Corollary 3.14 with $H = L_\lambda \leq \mathrm{Aut}(B(Q_\lambda))$. Then because $N_{L_\lambda}(O_2(E)) \geq S_{\delta\lambda}$, we get $\langle Z^{L_\lambda} \rangle \leq O_2(E)$. By Lemma 3.12, we have $\eta(E, O_2(E)) = \eta(E, Z) = 1$. Hence $\langle Z^{L_\lambda} \rangle$ is simultaneously normalized by L_λ and E, which contradicts Lemma 3.4. This concludes the proof of Theorem 3.20 in this case.

Next assume that $Z/C_Z(O^2(L))$ is an orthogonal S_5-module. In this situation we apply Lemma 3.16 to find a subgroup R of L such that $O_2(L) \leq R$, $B(Q_\lambda) \in Syl_2(R)$, $R/O_2(R) \cong L_2(2)$, and $\langle R, S \rangle = L$. Once again we notice that any nontrivial normal subgroup of R better not be characteristic in $B(Q_\lambda)$. Set $R_1 = RQ_\delta$. Then $O_2(R) = B(Q_\lambda) \cap O_2(R_1)$ is a normal subgroup of $B(Q_\lambda)Q_\delta$, where $B(Q_\lambda)Q_\delta$ has index 2 in $S_{\delta\lambda}$. Put $Z_0 = \langle \Omega_1(Z(B(Q_\lambda)))^R \rangle$. Then, by Lemma 3.12, $\eta(R, O_2(R)) = \eta(R, Z_0) = \eta(R, Z) = 1$, and so by the argument just presented in the prior case we have

$$(3.20.1) \quad \langle Z_0^{L_\lambda} \rangle \nleq O_2(R).$$

It follows from Corollary 3.14 and (3.20.1) that $N := N_{L_\lambda}(O_2(R))$ does not contain a Sylow 2-subgroup of L_λ. Since $B(Q_\lambda)Q_\delta \leq N$ we deduce that

(3.20.2) $B(Q_\lambda)Q_\delta \in Syl_2(N)$ and that $[L_\lambda : N] = 2n$, where n is odd.

Fix $y \in L_\lambda$ such that $Z_0^y \nleq O_2(R)$. Then, by Proposition 3.13, N normalizes $O_2(R^y)$ and $N_1 := \langle y \rangle N$ satisfies $[N_1 : N] = 2$. Notice that by (3.20.2), N_1 contains a Sylow 2-subgroup of L_λ. Choose $w \in N_1$ such that $y \in A = \langle w \rangle B(Q_\lambda)Q_\delta \in Syl_2(N_1)$. If $A = S_{\delta\lambda}$, then, as $w = ny$ for some $n \in N$, $O_2(R)^y = O_2(R)^w \geq O_2(L)^w = O_2(L) = Q_\delta \cap Q_\lambda \geq Z$. Thus, as $Z_0 = Z\Omega_1(Z(B(Q_\lambda)))$, $[Z_0, Z_0^y] = 1$, which is against our choice of y. Thus $A \neq S_{\delta\lambda}$. Set $D = \langle S_{\delta\lambda}, A \rangle$ and put $\overline{D} = D/B(Q_\lambda)Q_\delta$. Then, by (3.20.2), $\overline{D}$ is a dihedral group of order $2m$, where m is odd. Therefore, we have $O_2(R)^w = O_2(R)^y = O_2(R)^x$, where x is an element of odd order in $\overline{D}$. Now apply Corollary 3.14 with $H = D$. Then $x^2 \in N_D(O_2(R)) \geq B(Q_\lambda)Q_\delta$, whence, as $\langle x \rangle B(Q_\lambda)Q_\delta/B(Q_\lambda)Q_\delta$ has odd order $x \in N_D(O_2(R))$ and so $O_2(R) = O_2(R)^x = O_2(R)^y$, which contradicts the original choice of y. Therefore, $Z_\lambda \leq Z(L_\lambda)$ and the proof of the theorem is complete.

Our next result is also related to pushing-up.

LEMMA 3.21. *Suppose that* $(\delta, \lambda) \in E(\Gamma)$, $\delta \in O(L)$. *Then* $Q_\lambda Q_\delta/Q_\delta$ *contains a nontrivial subgroup normal in* $N_{L_\delta}(S_{\delta\lambda})/Q_\delta$.

Proof. Notice that, by Lemma 3.18(i), $Q_\delta \cap Q_\lambda$ is not normal in L_λ, whence $Q_\lambda \nleq Q_\delta$. Set $D := N_{L_\delta}(S_{\delta\lambda})$. Put $H = \langle D, L_\lambda \rangle$ and $R = \mathrm{core}_H(Q_\lambda)$, and suppose the result is false. First we consider the case when $\eta(L_\lambda, Q_\lambda/R) = 0$. Then $R \nleq Q_\delta$, for otherwise we obtain $Q_\delta \cap Q_\lambda \lhd L_\lambda$, which is contrary to Lemma 3.18(i). But then RQ_δ/Q_δ is a nontrivial subgroup of $Q_\lambda Q_\delta/Q_\delta$ which is normalized by D. Thus $\eta(L_\lambda, Q_\lambda/R) \neq 0$, which yields $1 \neq \Omega_1(Z(S_{\delta\lambda}/R)) \leq \Omega_1(Z(Q_\lambda/R))$ and then, as $\Omega_1(Z(S_{\delta\lambda}/R)) \lhd D/R$, $\eta(L_\lambda, \Omega_1(Z(Q_\lambda/R))) \geq 1$. Since $J(S_{\delta\lambda}/R) \nleq Q_\lambda/R$ we conclude that $\Omega_1(Z(Q_\lambda/R))$ admits an FF-action by L_λ/Q_λ; however, this contradicts Theorem 2.15(iii). Therefore the lemma holds.

The last result of this section is designed so as to deal with some of the exceptional S_5 problems that we will encounter later on.

LEMMA 3.22. *Assume that* $(\delta, \lambda) \in E(\Gamma)$, $L_\delta/Q_\delta \cong S_5$, *and* $b \geq 2$. *Let* Z *be a minimal normal subgroup of* L_δ *contained in* Z_δ *and assume that* $\eta(L_\delta, Z) = 1$ *with* $C_Z(L_\delta) = 0$. *Put* $V = \langle Z^{L_\lambda} \rangle$. *Then one of the following holds*:

(i) $Q_\delta Q_\lambda \leq F_\lambda = C_{L_\lambda}([Z, Q_\lambda]) \underset{2}{<} L_\lambda$ *and* $[V, Q_\lambda] = [Z, Q_\lambda] \cong E(2^2)$;

(ii) $\eta(L_\lambda, V) \geq 2$ *and* $\eta(L_\lambda, [V, Q_\lambda]) \neq 0$.

Proof. Put $Y = \operatorname{core}_{L_\lambda}(Z)$. Note that the assumptions on Z imply that Z is either a natural or an orthogonal L_δ/Q_δ-module (and so $|Z| = 2^4$). Supposing that $\eta(L_\lambda, V) \leq 1$ or $\eta(L_\lambda, [V, Q_\lambda]) = 0$ we show that (i) holds. By Lemma 3.7(i), $\eta(L_\lambda, V/[V, Q_\lambda]) \geq 1$, and so in either case we obtain $[V, Q_\lambda] = [Z, Q_\lambda] \leq Y$. If $Q_\lambda \leq Q_\delta$, then we have a contradiction to Lemma 3.18. So we have that $Q_\lambda \nleq Q_\delta$. Also observe, as $S_{\delta\lambda}/Q_\delta \cong D_8$ and $L_\lambda/Q_\lambda \in \mathcal{S}$, that $Q_\delta \nleq Q_\lambda$. Suppose $|Y| \geq 2^3$, and let $g \in L_\lambda$. Now $Q_\delta \cap Q_\lambda$ centralizes $Y = Y^g$ $(= [Z^g, S_{\delta(\lambda\cdot g)}])$ and so Lemma 2.3(iv) forces $Q_\delta \cap Q_\lambda \leq Q_\delta^g$. Hence $Q_\delta \cap Q_\lambda = K_\lambda \lhd L_\lambda$, which is again contrary to Lemma 3.18. Therefore $|Y| \leq 2^2$. Using Lemma 2.3(iv) (as $Q_\lambda \nleq Q_\delta$) gives $\|[Z, Q_\lambda]\| \neq 2$ and, therefore, $Y = [Z, Q_\lambda] \cong E(2^2)$. Now it follows from Lemma 2.3(ii), (iv)(a, b) and (vi)(a, b) that $Q_\delta Q_\lambda$ centralizes Y, and hence $\langle (Q_\delta Q_\lambda)^{L_\lambda} \rangle$ centralizes Y. Since $S_{\delta\lambda}$ cannot centralize Y (by Lemma 2.3(v)(b) and (vi)(b)) and recalling that $Q_\delta \nleq Q_\lambda$, we infer that $F_\lambda \leq \langle (Q_\delta Q_\lambda)^{L_\lambda} \rangle \neq L_\lambda$. So, from [$\mathbb{A}$], $[L_\lambda : F_\lambda] = 2$ and we have (i), and this completes the proof of Lemma 3.22.

4. NONCOMMUTING CRITICAL PAIRS

In this section we assume that the following hypothesis is fulfilled:

HYPOTHESIS 4.1. *Hypothesis* 3.1 *holds and for* $(\alpha, \alpha') \in \mathcal{C}$, $[Z_\alpha, Z_{\alpha'}] \neq 1$.

The goal of this section is the proof of the following theorem:

THEOREM 4.2. *Suppose that Hypothesis* 4.1 *is satisfied. Then* $L_\alpha/Q_\alpha \cong L_2(2)$, Q_β *is extraspecial of* $+$*-type, and one of the following holds*:

(i) $L_\beta/Q_\beta \cong 3\,\mathrm{Aut}(M_{22})$, $Q_\beta \cong 2_+^{1+12}$, $\eta(L_\alpha, Q_\alpha) = 6$, *and* L_α *has a chief series described by* $2^{2+1_5+2_5+1_3}L_2(2)$.

(ii) $L_\beta/Q_\beta \cong Co_2$, $Q_\beta \cong 2_+^{1+22}$, $\eta(L_\alpha, Q_\alpha) = 11$, *and* L_α *has a chief series described by* $2^{2+1_{10}+2_{10}+1_8}L_2(2)$.

(iii) $L_\beta/Q_\beta \cong Co_1$, $Q_\beta \cong 2_+^{1+24}$, $\eta(L_\alpha, Q_\alpha) = 12$, *and* L_α *has a chief series described by* $2^{2+1_{11}+2_{11}+1_{10}}L_2(2)$.

We prove the theorem via a series of lemmas. We begin with

LEMMA 4.3. *The following hold*:

(i) b *is even*;

(ii) $L_\alpha/Q_\alpha \cong L_2(q_\alpha)$ *or* S_5;

(iii) $Z_\alpha/C_{Z_\alpha}(O^2(L_\alpha))$ *is a natural* L_α/Q_α*-module or* $L_\alpha/Q_\alpha \cong S_5$ *and* $Z_\alpha/C_{Z_\alpha}(O^2(L_\alpha))$ *is an orthogonal module*;

(iv) *$Z_\alpha Q_{\alpha'}/Q_{\alpha'}$ is an offending subgroup on $Z_{\alpha'}$ and $Z_{\alpha'}Q_\alpha/Q_\alpha$ is an offending subgroup on Z_α; moreover, $|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = |Z_{\alpha'}Q_\alpha/Q_\alpha| = q_\alpha$ or $Z_\alpha/C_{Z_\alpha}(O^2(L_\alpha))$ is an orthogonal S_5-module and $|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = |Z_{\alpha'}Q_\alpha/Q_\alpha| = 2$.*

Proof. Suppose that $(\alpha, \alpha') \in \mathcal{C}$. Then, as $(\alpha', \alpha) \in \mathcal{C}$, by Proposition 3.6(ii), we may assume, without loss of generality, that $|Z_{\alpha'}Q_\alpha/Q_\alpha| \geq |Z_\alpha Q_{\alpha'}/Q_{\alpha'}|$. Thus, by Lemma 3.5(ii) and Proposition 3.6(i),

$$\left[Z_\alpha : C_{Z_\alpha}(Z_{\alpha'})\right] = [Z_\alpha : Z_\alpha \cap Q_{\alpha'}] = |Z_\alpha Q_{\alpha'}/Q_{\alpha'}| \leq |Z_{\alpha'}Q_\alpha/Q_\alpha|. \quad (*)$$

Thus Z_α is an *FF*-module for L_α/Q_α. Applying Theorem 2.15(iii) and Lemmas 2.3, 2.32, and 2.38, we deduce that $L_\alpha/Q_\alpha \cong L_2(q_\alpha)$ or S_5 and that $Z_\alpha/C_{Z_\alpha}(O^2(L_\alpha))$ is a natural L_α/Q_α-module or an orthogonal S_5-module. We now use Lemmas 2.3 and 2.32 and $(*)$, to find that $|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = |Z_{\alpha'}Q_\alpha/Q_\alpha|$. Thus the configuration is symmetric, and so $L_{\alpha'}/Q_{\alpha'} \cong L_2(q_\alpha)$, or S_5; in particular, b is even and we have proven all parts of the lemma.

LEMMA 4.4. *$Z_\beta \leq Z(L_\beta)$.*

Proof. Suppose that $Z_\beta \nleq Z(L_\beta)$ and pick $\alpha - 1 \in \Delta(\alpha)$ such that $\langle S_{\alpha-1\alpha}, Z_{\alpha'}\rangle = L_\alpha$. Then, by Lemma 4.3(ii), $Z_{\alpha-1} \leq Q_{\alpha'-1}$. Assume that $Z_{\alpha-1}Z_\alpha \leq Z_\alpha Q_{\alpha'}$. Then

$$[Z_{\alpha-1}Z_\alpha, Z_{\alpha'}] = [Z_\alpha, Z_{\alpha'}] \leq Z_\alpha.$$

Therefore $Z_{\alpha-1}Z_\alpha \lhd L_\alpha$. But then $C_{Q_\alpha}(Z_{\alpha-1}Z_\alpha) = Q_{\alpha-1} \cap Q_\alpha$ is normal in L_α, so Theorem 3.20 implies that $Z_\beta \leq Z(L_\beta)$, which is against our assumption. Hence $Z_{\alpha-1}Z_\alpha Q_{\alpha'}/Q_{\alpha'} > Z_\alpha Q_{\alpha'}/Q_{\alpha'}$. Since $b \geq 2$, $Z_{\alpha-1}Z_\alpha Q_{\alpha'}/Q_{\alpha'}$ is elementary abelian and hence we obtain, using Lemma 4.3(iv), $L_\alpha/Q_\alpha \cong S_5 \cong L_{\alpha'}/Q_{\alpha'}$, $|Z_{\alpha-1}Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = 2^2$, $|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = 2$, and $Z_{\alpha'}/C_{Z_{\alpha'}}(O^2(L_{\alpha'}))$ is an orthogonal S_5-module. From Lemma 2.3(v)(a), we get that $Z_{\alpha'} \cap Q_\alpha = C_{Z_{\alpha'}}(Z_\alpha)$ does not centralize $Z_{\alpha-1}$ and so $Z_{\alpha'} \cap Q_\alpha \nleq Q_{\alpha-1}$. Since $[Z_{\alpha-1} : C_{Z_{\alpha-1}}(Z_{\alpha'} \cap Q_\alpha)] \leq 2^2$, this contradicts Theorem 2.15(v). Hence the result is proven.

The next two results are easy consequences of Lemmas 4.3 and 4.4.

LEMMA 4.5. *Suppose that $L_\alpha/Q_\alpha \cong L_2(q_\alpha)$. Then*

(i) $S_{\alpha\beta} = Q_\alpha Q_\beta$;

(ii) *Z_α is a natural L_α/Q_α-module*;

(iii) $[Z_\alpha, Z_{\alpha'}] = [Z_\alpha, Q_\beta] = [Z_{\alpha'}, Q_{\alpha'-1}] = Z_\beta = Z_{\alpha'-1}$.

Proof. From Lemma 4.4, $Z_\beta = \Omega_1(Z(S_{\alpha\beta}))$ and hence $C_{Z_\alpha}(O^2(L_\alpha)) = 1$. Thus, by Lemma 4.3, Z_α and $Z_{\alpha'}$ are natural modules for L_α/Q_α and $L_{\alpha'}/Q_{\alpha'}$, respectively. Now (i), (ii), and (iii) follow from Lemma 4.3(iv) and Lemma 2.35.

LEMMA 4.6. *Suppose that $L_\alpha/Q_\alpha \cong S_5$. Then Z_α is either a natural or an orthogonal S_5-module. Furthermore, if $|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = 4$, then $[Z_\alpha, Z_{\alpha'}] \geq Z_{\alpha'-1}Z_\beta$.*

Proof. Again we have $C_{Z_\alpha}(O^2(L_\alpha)) = 1$, and the lemma follows using Lemma 4.3(iii) and Lemma 2.3(v)(a) and (vi)(b).

LEMMA 4.7. *Suppose that $b \geq 4$ and Z_α is a natural L_α/Q_α-module. Then $U_\alpha \leq S_{\alpha'-2\alpha'-1} \cap S_{\alpha'-2\alpha'-3} \leq L_{\alpha'-1}$ and $U_{\alpha'} \leq S_{\alpha+2\beta} \cap S_{\alpha+2\alpha+3} \leq L_\beta$.*

Proof. From Lemmas 4.5, 4.3(iv), and 4.6, $Z_\beta Z_{\alpha'-1} \leq [Z_\alpha, Z_{\alpha'}] \leq Z_\alpha \cap Z_{\alpha'}$ (note that when $L_\alpha/Q_\alpha \cong S_5$ we have $|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = 4$ as Z_α is a natural module). Hence, as $b \geq 4$, $U_{\alpha'}$ centralizes Z_β and then Lemma 2.34 implies

$$U_{\alpha'} \leq C_{L_{\alpha+2}}\big(Z_\beta\big) \leq S_{\alpha+2\beta} \leq L_\beta .$$

Similarly, we deduce that $U_\alpha \leq S_{\alpha'-2\alpha'-1} \cap S_{\alpha'-2\alpha'-3} \leq L_{\alpha'-1}$.

LEMMA 4.8. *Suppose that $b \geq 4$. Then $q_\alpha = 2$ or $L_\alpha/Q_\alpha \cong S_5$.*

Proof. We suppose to the contrary that $q_\alpha \geq 4$ and $L_\alpha/Q_\alpha \not\cong S_5$. Then $L_\alpha/Q_\alpha \cong L_2(q_\alpha)$ with $q_\alpha \geq 4$, by Lemma 4.3(ii). Note that $Z_\beta = Z_{\alpha'-1}$ (by Lemma 4.5) and $C_{Z_{\alpha'}}(Z_\alpha) = Z_{\alpha'-1}$ imply that $V_\beta \cap Z_{\alpha'} = Z_{\alpha'-1} = Z_\beta$. Also from Lemma 4.5 we deduce that $[V_\beta, Q_\beta] = Z_\beta$ and that $[U_\alpha, U_\alpha] \leq Z_\alpha$. Employing Lemma 4.7 gives

$$\big[V_\beta, U_{\alpha'}, U_{\alpha'}\big] \leq V_\beta \cap [U_{\alpha'}, U_{\alpha'}] \leq V_\beta \cap Z_{\alpha'} = Z_\beta .$$

Thus $U_{\alpha'}$ acts quadratically on V_β/Z_β. Since $[Z_{\alpha'} : C_{Z_{\alpha'}}(Z_\alpha)] = q_\alpha$ by Lemma 4.3(iv), Lemma 3.7(ii) forces $|U_{\alpha'}Q_\beta/Q_\beta| \geq 2^2$. Consulting Theorem 2.15(i) gives $|U_{\alpha'}Q_\beta/Q_\beta| \leq 2^4$ and hence, as $q_\alpha \geq 4$, Lemma 2.31 gives

(4.8.1) either (i) $\eta(L_\alpha, U_\alpha/Z_\alpha) = 1$ or (ii) $\eta(L_\alpha, U_\alpha/Z_\alpha) = 2$, $q_\alpha = 4$, $F_\beta/D_\beta \cong 3M_{22}$, and $|U_{\alpha'}Q_\beta/Q_\beta| = 2^4$.

Suppose that (4.8.1)(i) pertains. Then $[V_\beta, Q_{\alpha+2}]Z_{\alpha+2} = [V_{\alpha+3}, Q_{\alpha+2}]Z_{\alpha+2}$, which yields $[V_\beta, Q_{\alpha+2}] \leq V_{\alpha+3} \leq Q_{\alpha'+1}$ for all $\alpha' + 1 \in \Delta(\alpha')$. Hence

$$\big[\big[V_\beta, Q_{\alpha+2}\big], U_{\alpha'}\big] \leq Z_{\alpha'} \cap V_\beta = Z_\beta .$$

Thus, as $Q_{\alpha+2}Q_\beta = S_{\beta\alpha+2}$, we have a contradiction to Proposition 2.16(ii).

Finally, considering (4.8.1)(ii) we have here that $[V_\beta, Q_{\alpha+2}, Q_{\alpha+2}]Z_{\alpha+2} = [V_{\alpha+3}, Q_{\alpha+2}, Q_{\alpha+2}]Z_{\alpha+2}$, from which we similarly infer that

$$\big[\big[V_\beta, Q_{\alpha+2}, Q_{\alpha+2}\big], U_{\alpha'}\big] \le Z_\beta.$$

This situation is impossible by Proposition 2.12(iv) and so we have shown that either $q_\alpha = 2$ or $L_\alpha/Q_\alpha \cong S_5$.

LEMMA 4.9. *Suppose that $b \ge 4$. Then $L_\alpha/Q_\alpha \cong S_5$ and Z_α is an orthogonal S_5-module.*

Proof. Suppose the lemma is false. Thus, by Lemmas 4.3(ii) and 4.8, our job here is to show that $L_\alpha/Q_\alpha \cong L_2(2)$ as well as the situation when $L_\alpha/Q_\alpha \cong S_5$ and Z_α is a natural S_5-module lead to a contradiction. So choose $\alpha - 1 \in \Delta(\alpha)$ such that $\langle S_{\alpha-1\alpha}, Z_{\alpha'}\rangle = L_\alpha$. From Lemma 4.7 we have

(4.9.1) $V_{\alpha-1} \le S_{\alpha'-2\alpha'-3} \cap S_{\alpha'-2\alpha'-1} \le L_{\alpha'-1}$.

Assume that $V_{\alpha-1} \le G_{\alpha'}$. Then, by Lemma 4.3(iv), $V_{\alpha-1}Q_{\alpha'} = Z_\alpha Q_{\alpha'}$, which implies that $[V_{\alpha-1}, Z_{\alpha'}] = [Z_\alpha, Z_{\alpha'}] \le Z_\alpha \le V_{\alpha-1}$. But then $V_{\alpha-1}$ is normalized by $\langle S_{\alpha-1\alpha}, Z_{\alpha'}\rangle = L_\alpha$, a contradiction. Thus $V_{\alpha-1} \not\le G_{\alpha'}$.

Also, if $[V_{\alpha-1}, V_{\alpha'-1} \cap Q_{\alpha-1}] \ne 1$, then, by Lemma 2.3(ii) and (v)(a) and Lemma 2.35, for some $\alpha - 2 \in \Delta(\alpha - 1)$, $[V_{\alpha-1}, V_{\alpha'-1} \cap Q_{\alpha-1}] \ge [Z_{\alpha-2}, V_{\alpha'-1} \cap Q_{\alpha-1}] \ge Z_{\alpha-1}$, which gives the contradiction $Z_{\alpha-1} \lhd \langle S_{\alpha-1\alpha}, Z_{\alpha'}\rangle = L_\alpha$. Thus

(4.9.2) (i) $V_{\alpha-1} \not\le Q_{\alpha'-1}$ and (ii) $[V_{\alpha-1}, V_{\alpha'-1} \cap Q_{\alpha-1}] = 1$.

Our next aim is to prove

(4.9.3) If $F_\beta/D_\beta \not\cong 3M_{22}$, then $\eta(L_\beta, V_\beta) = 1$ and $|V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}| = 2$.

Suppose that $F_\beta/D_\beta \not\cong 3M_{22}$. Using (4.9.1), $V_{\alpha'-1} \cap Q_\alpha$ acts quadratically on $V_{\alpha-1}$, so Theorem 2.15(i) implies that $|(V_{\alpha'-1} \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}| \le 2^2$. This, together with (4.9.2), yields

$$\big[V_{\alpha'-1} : C_{V_{\alpha\setminus-1}}(V_{\alpha-1})\big] \le \begin{cases} 2^3 & \text{if } L_\alpha/Q_\alpha \cong L_2(2), \\ 2^4 & \text{if } L_\alpha/Q_\alpha \cong S_5, \end{cases} \qquad (*)$$

Now Theorem 2.15(v) implies that $\eta(L_\beta, V_\beta) = 1$. Furthermore, in the case when $L_\alpha/Q_\alpha \cong S_5$, Lemma 3.22 implies that $Q_{\alpha'-1}Q_{\alpha'-2} \le F_{\alpha'-1}$. Now we assume that $|V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}| \ge 2^2$ and argue for a contradiction. First we claim that $\overline{V}_{\alpha'-1} = V_{\alpha'-1}/[V_{\alpha'-1}, Q_{\alpha'-1}]$ is a quadratic $L_{\alpha'-1}/D_{\alpha'-1}$-module. Indeed if $|(V_{\alpha'-1} \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}| = 2^2$, then we are done by (4.9.1). Hence we may assume that $|(V_{\alpha'-1} \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}| \leq 2$. But then $[V_{\alpha'-1} : C_{V_{\alpha'-1}}(V_{\alpha-1})] \leq 2^3$ by (4.9.2)(ii), and the claim follows from Theorem 2.15(v). If $L_\alpha/Q_\alpha \cong L_2(2)$, then $(*)$, Theorem 2.15(ii), Proposition 2.11(ii), and $|V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}| \geq 2^2$ yields a contradiction. While $L_\alpha/Q_\alpha \cong S_5$ implies, using $(*)$ and Theorem 2.15(ii), that $L_{\alpha'-1}/D_{\alpha'-1} \cong \mathrm{Aut}(M_{22})$. Recalling that $Q_{\alpha'-2}Q_{\alpha'-1} \leq F_{\alpha'-1}$, we see that $V_{\alpha-1} \nleq Q_{\alpha'-2}$ by Proposition 2.11(ii). Hence, there exists $\alpha - 2 \in \Delta(\alpha - 1)$ such that $(\alpha - 2, \alpha' - 2) \in \mathcal{C}$. Therefore, as $Z_{\alpha'-2}$ is a natural $L_{\alpha'-2}/Q_{\alpha'-2}$-module,

$$\begin{aligned} V_{\alpha-1}Q_{\alpha'-2}/Q_{\alpha'-2} &= S_{\alpha'-3\alpha'-2}/Q_{\alpha'-2} \cap O^2(L_{\alpha'-2})/Q_{\alpha'-2} \\ &= Q_{\alpha'-3}Q_{\alpha'-2}/Q_{\alpha'-2}. \end{aligned}$$

Hence, employing (4.9.1),

$$\begin{aligned} V_{\alpha-1}Q_{\alpha'-2}/Q_{\alpha'-2} &= S_{\alpha'-2\alpha'-1}/Q_{\alpha'-2} \cap S_{\alpha'-3\alpha'-2}/Q_{\alpha'-2} \\ &= Q_{\alpha'-1}Q_{\alpha'-2}/Q_{\alpha'-2}, \end{aligned}$$

and $V_{\alpha-1} \leq Q_{\alpha'-1}Q_{\alpha'-2} \leq F_{\alpha'-1}$, which is a contradiction, so verifying (4.9.3).

(4.9.4) If $\eta(L_\beta, V_\beta) = 1$, then $|V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}| \neq 2$.

Suppose that $\eta(L_\beta, V_\beta) = 1$ and $|V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}| = 2$. Then either $L_\alpha/Q_\alpha \cong L_2(2)$ and $[V_{\alpha'-1}, Q_{\alpha'-1}] = Z_{\alpha'-1} \cong E(2)$, or $L_\alpha/Q_\alpha \cong S_5$ and $[V_{\alpha'-1}, Q_{\alpha'-1}] \cong E(2^2)$, by Lemma 3.22. Since $[V_{\alpha'-1} : V_{\alpha'-1} \cap Q_\alpha] \leq 2^2$, (4.9.2) and Theorem 2.15(v) force $V_{\alpha'-1} \cap Q_\alpha \nleq Q_{\alpha-1}$. Let $1 \neq t \in (V_{\alpha'-1} \cap Q_\alpha) \backslash Q_{\alpha-1}$. Then we have $|[V_{\alpha-1} \cap Q_{\alpha'-1}, t]| \leq |[V_{\alpha'-1}, Q_{\alpha'-1}]| \leq 2^2$. So, since $[V_{\alpha-1} : V_{\alpha-1} \cap Q_{\alpha'-1}] = 2$, $[V_{\alpha-1} : C_{V_{\alpha-1}}(t)] \leq 2^3$. Hence Theorem 2.15(v), Proposition 2.11(iv) and (v), and Lemma 3.22 imply that $L_\beta/Q_\beta \cong \mathrm{Aut}(M_{22})$ and $L_\alpha/Q_\alpha \cong L_2(2)$. But then $|[V_{\alpha-1} \cap Q_{\alpha'-1}, t]| \leq |Z_{\alpha'-1}| = 2$, so $[V_{\alpha-1} : C_{V_{\alpha-1}}(t)] \leq 2^2$, which is against Theorem 2.15(v). Thus (4.9.4) holds.

Putting (4.9.3) and (4.9.4) together and observing that, by (4.9.2)(ii), $[V_{\alpha'-1} : C_{V_{\alpha}\backslash -1}(V_{\alpha-1})] \leq 2^6$ when $F_{\alpha-1}/D_{\alpha-1} \cong 3M_{22}$ we have, from Theorem 2.15(v),

(4.9.5) $F_\beta/D_\beta \cong 3M_{22}$, and $\eta(L_\beta, V_\beta) = 1$.

Combining (4.9.2)(ii), (4.9.5), and Theorem 2.15(v), we see that $|(V_{\alpha'-1} \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}| \geq 2^2$ and so, by (4.9.1), the noncentral $L_{\alpha-1}$-chief factor in $V_{\alpha-1}$ is a quadratic module. Since $|V_{\alpha-1}Q_{\alpha'-1}/Q_{\alpha'-1}| \geq 2^2$ by (4.9.4) and (4.9.5), Proposition 2.12(ii) demands that $|(V_{\alpha'-1} \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}| = 2^4$ and $|V_{\alpha'-1}Q_\alpha/Q_\alpha| = 2^2$. Thus $L_\alpha/Q_\alpha \cong S_5$. By Lemma 4.3(iv) we have

$[S_{\alpha'-2\alpha'-1} : Q_{\alpha'-1}Q_{\alpha'-2}] \leq 2$ and so appealing to Lemma 3.22 we deduce that $Q_{\alpha'-1}Q_{\alpha'-2} \in Syl_2(F_{\alpha'-1})$. Employing Lemma 3.22 again gives

$$E(2^2) \cong [Z_{\alpha'}, Q_{\alpha'-1}] = [V_{\alpha'-1}, Q_{\alpha\backslash-1}] = [Z_{\alpha'-2}, Q_{\alpha'-1}],$$

and hence $[Z_\alpha, Z_{\alpha'}] = Z_{\alpha'} \cap Z_{\alpha'-2} = [Z_{\alpha'}, Q_{\alpha'-1}]$. Now, by Lemma 4.7, $U_\alpha \leq L_{\alpha'-1}$, and so

$$\begin{aligned}[V_{\alpha'-1}, U_\alpha, U_\alpha] &\leq [U_\alpha, U_\alpha] \cap V_{\alpha'-1} \leq Z_\alpha \cap V_{\alpha'-1} \\ &= [Z_\alpha, Z_{\alpha'}] = [Z_{\alpha'}, Q_{\alpha'-1}] = [V_{\alpha'-1}, Q_{\alpha'-1}].\end{aligned}$$

Thus U_α acts quadratically on $V_{\alpha'-1}/[V_{\alpha'-1}, Q_{\alpha'-1}]$ and hence $|U_\alpha Q_{\alpha'-1}/Q_{\alpha'-1}| \leq 2^4$ by Proposition 2.12(i). Therefore, since $[U_\alpha, Z_\alpha] = 1$, $[U_\alpha : C_{U_\alpha}(Z_{\alpha'})] \leq 2^6$ and so, as $|Z_{\alpha'}Q_\alpha/Q_\alpha| = 2^2$, $\eta(L_\alpha, U_\alpha) \leq 3$. Consequently,

$$[U_{\alpha'-2}, Q_{\alpha'-2}; n] = [V_{\alpha'-3}, Q_{\alpha'-2}; n] = [V_{\alpha'-1}, Q_{\alpha'-2}; n],$$

where $n = \eta(L_\alpha, U_\alpha) - 1$. Because, for $(\delta, \delta') \in \mathcal{C}$, we have $[Z_\delta, Z_{\delta'}] \leq Z_\delta \cap Z_{\delta+2}$ we obtain

$$[[V_{\alpha'-3}, Q_{\alpha'-2}; n], U_\alpha] \leq V_{\alpha'-1} \cap Z_\alpha = [Z_\alpha, Z_{\alpha'}] \leq [V_{\alpha'-1}, Q_{\alpha'-1}].$$

Note that $V_{\alpha'-1}/[V_{\alpha'-1}, Q_{\alpha'-1}]$ is an $F_{\alpha'-1}/D_{\alpha'-1}$-module with $Q_{\alpha'-2}$ acting as a Sylow 2-subgroup of $F_{\alpha'-1}/D_{\alpha'-1}$. From $U_\alpha \geq V_{\alpha-1}$, $|U_\alpha Q_{\alpha'-1}/Q_{\alpha'-1}| \geq 2^2$ and so if $n = 1$ we have a contradiction to Proposition 2.16(ii). In the case $n = 2$ (so $\eta(L_\alpha, U_\alpha) = 3$), we then have $|U_\alpha Q_{\alpha'-1}/Q_{\alpha'-1}| = 2^4$, which is at variance with Proposition 2.12(iv).

This completes the proof of the lemma.

LEMMA 4.10. $b = 2$.

Proof. Suppose that G is a counterexample to Lemma 4.10 and let $(\alpha, \alpha') \in \mathcal{C}$. Then, by Lemma 4.9, $L_\alpha/Q_\alpha \cong S_5$ and Z_α is an orthogonal S_5-module. If $\tau \in O(\mathcal{L})$ and $(\tau, \zeta) \in E(\Gamma)$, then the unique subgroup of L_τ properly containing $S_{\tau\zeta}$ is denoted by $M_{\tau\zeta}$; note that $E(2^2) \cong O_2(M_{\tau\zeta})/Q_\tau \leq O^2(L_\tau/Q_\tau)$ and $M_{\tau\zeta}/O_2(M_{\tau\zeta}) \cong L_2(2)$. Our first result follows from Lemmas 2.3(iii) and 4.3(iv).

(4.10.1) $Z_{\alpha'} \nleq O_2(M_{\alpha\beta})$.

Now we define $H = \langle M_{\alpha\beta}, L_\beta \rangle$ and $Q_H = \mathrm{core}_H(L_\beta \cap M_{\alpha\beta})$.

(4.10.2) $\eta(L_\beta, Q_\beta/Q_H) \geq 1$.

Suppose that $\eta(L_\beta, Q_\beta/Q_H) = 0$. Then

$$Q_\beta \cap O_2(M_{\alpha\beta}) = Q_\beta \cap O_2(M_{\alpha+2\beta}) \geq Q_\beta \cap Q_{\alpha+2} \geq Z_{\alpha'},$$

which contradicts (4.10.1).

(4.10.3) One of the following holds:

(i) $\mathrm{core}_{L_\beta}(O_2(M_{\alpha\beta})) = Q_H$;

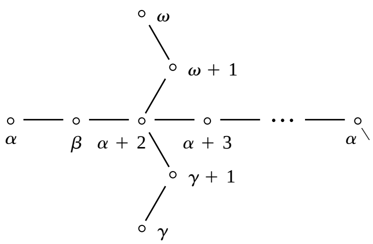
(ii) $\mathrm{core}_{L_\gamma}(O_2(M_{\alpha+2\beta})) \cap \mathrm{core}_{L_\beta}(O_2(M_{\alpha+2\beta})) = Q_H$, where γ is any vertex in $\Delta(\alpha+2)\backslash\{\beta\}$ with $M_{\alpha+2\gamma} = M_{\alpha+2\beta}$.

If $\eta(M_{\alpha\beta}, O_2(M_{\alpha\beta})/Q_H) = 0$, then clearly $\mathrm{core}_{L_\beta}(O_2(M_{\alpha\beta}))$ is normalized simultaneously by L_β and $M_{\alpha\beta}$, whence we obtain (i). While $\eta(M_{\alpha\beta}, O_2(M_{\alpha\beta})/Q_H) \neq 0$ together with (4.10.2) yield the hypothesis of Theorem B, and then (ii) holds by Theorem B.

Since, by Lemma 3.18, $Q_\alpha \cap Q_\beta$ is not normal in L_β, the following fact follows from Lemma 2.3(iv).

(4.10.4) $Y_\beta := \mathrm{core}_{L_\beta}(Z_\alpha) = [Z_\alpha, S_{\alpha\beta}; n] \cong 2^{4-n}$, for $n = 2$ or 3.

For the next few steps we will be concerned with the following subgraph of Γ:



where $M_{\alpha+2\beta} = M_{\alpha+2\omega+1} = M_{\alpha+2\gamma+1}$.

(4.10.5) Suppose that $|Y_\beta| = 2^2$. Then, up to a relabelling of vertices, we have (i) $Z_{\alpha'} \leq \mathrm{core}_{L_{\gamma+1}}(O_2(M_{\alpha+2\gamma+1}))$; and (ii) $(\alpha, \alpha'), (\omega, \alpha') \in \mathcal{C}$, $|Z_\omega Q_{\alpha'}/Q_{\alpha'}| = |Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = 2$, and $Z_\alpha Q_{\alpha'}/Q_{\alpha'} \neq Z_\omega Q_{\alpha'}/Q_{\alpha'}$ both act as transvections on $Z_{\alpha'}$.

Suppose that there exist $\tau_1, \tau_2 \in \Delta(\beta) \cup \Delta(\omega+1) \cup \Delta(\gamma+1)$ with $d(\tau_1, \tau_2) = 4$ and $1 \neq [Z_{\tau_1}, Z_{\alpha'}] \leq [Z_{\tau_2}, Z_{\alpha'}]$. Without loss of generality we assume that $\tau_1 \in \Delta(\beta)$ and $\tau_2 \in \Delta(\gamma+1)$. Then, by Lemma 2.3(v)(e) (applied at $\alpha+2$),

$$[Z_{\tau_1}, Z_{\alpha'}] \leq [Z_{\alpha+2}, S_{\alpha+2\beta}; 2] \cap [Z_{\alpha+2}, S_{\alpha+2\gamma+1}; 2] = Z_\beta = Z_{\gamma+1},$$

which is against Lemma 2.3(iv) (applied at τ_1).

Now suppose that we find $(\alpha, \alpha'), (\omega, \alpha'), (\gamma, \alpha') \in \mathcal{C}$ (see the diagram following (4.10.4)). Then, since $b \geq 4$ and $S_{\alpha'-1\alpha'}/Q_{\alpha'}$ contains exactly two transvections, we have to conclude that there exists a pair $\{\tau_1, \tau_2\} \subseteq \{\alpha, \omega, \gamma\}$ with $1 \neq [Z_{\tau_1}, Z_{\alpha'}] \leq [Z_{\tau_2}, Z_{\alpha'}]$, which we have already shown is impossible. It follows that we may choose notation so that $Z_{\alpha'} \leq \operatorname{core}_{L_{\gamma+1}}(O_2(M_{\alpha+2\gamma+1}))$ and (i) is true. Fix this γ. Now if, for all $\omega \in \Delta(\omega+1)$ $Z_{\alpha'} \leq Q_\omega$, then, by (4.10.3)(ii),

$$Z_{\alpha'} \leq \operatorname{core}_{L_{\gamma+1}}\big(O_2(M_{\alpha+2\beta})\big) \cap \operatorname{core}_{L_{\omega+1}}\big(O_2(M_{\alpha+2\beta})\big) = Q_H \leq O_2(M_{\alpha\beta}),$$

which is against (4.10.1). Thus (4.10.5)(ii) holds.

(4.10.6) $|Y_\beta| = 2$.

Suppose that $|Y_\beta| = 2^2$. Then (4.10.5)(i) and Theorem B imply that

$$1 \neq Z_{\alpha'}Q_H/Q_H \leq \operatorname{core}_{L_{\gamma+1}}\big(O_2(M_{\alpha+2\gamma+1})\big)/Q_H \cong E(2).$$

Hence, by (4.10.5)(ii),

$$Z_{\alpha'} \cap Q_\alpha = Z_{\alpha'} \cap Q_H = Z_{\alpha'} \cap Q_\omega.$$

But then the transvections $Z_\alpha Q_{\alpha'}/Q_{\alpha'}$ and $Z_\omega Q_{\alpha'}/Q_{\alpha'}$ centralize the same hyperplane in $Z_{\alpha'}$, which contradicts (4.10.5)(ii) and Lemma 2.3(v)(a). Hence $|Y_\beta| = 2$.

Now define $H_\beta := \langle [Z_\alpha, S_{\alpha\beta}; 2]^{L_\beta} \rangle$. Then one of the consequence of (4.10.6) is that $\eta(L_\beta, H_\beta/Z_\beta) \geq 1$. Because of Lemma 2.3(iv), $Q_\beta Q_\alpha/Q_\alpha$ is either the quadratic fours group on Z_α or $S_{\alpha\beta}/Q_\alpha$ and so $[Z_\alpha, Q_\beta] \geq [Z_\alpha, S_{\alpha\beta}; 2]$, whence $[V_\beta, Q_\beta] \geq H_\beta$. Therefore, $\eta(L_\beta, V_\beta) \geq 2$ and $Z_\alpha \nleq H_\beta$.

We next put $X_\alpha = \langle H_\beta^{L_\alpha} \rangle$.

(4.10.7) $\eta(L_\alpha, X_\alpha) \geq 3$.

Suppose that $\eta(L_\alpha, X_\alpha) \leq 2$. Then, using Lemma 3.7(ii), $[X_\alpha, Q_\alpha]Z_\alpha = [H_\beta, Q_\alpha]Z_\alpha \lhd L_\alpha$ and, as $Z_\alpha \nleq H_\beta$, $[H_\beta, Q_\alpha, Q_\alpha] = 1$. Hence Q_α acts quadratically on H_β/Z_β. Since $[S_{\alpha\beta} : Q_\alpha Q_\beta] \leq 2$, we have a contradiction to Theorem 2.15(i).

(4.10.8) The contradiction.

Pick $\alpha - 2 \in \Delta(\alpha-1)\backslash\{\alpha\}$ and set $J_{\alpha-2\alpha-1} = [Z_{\alpha-2}, S_{\alpha-2\alpha-1}; 2]$. So $J_{\alpha-2\alpha-1}$ is a typical generator of X_α. Notice that, by Lemmas 4.3(iv) and 2.3(v)(a), $[J_{\alpha-2\alpha-1}, Z_{\alpha'-2}] = 1$, and thus $X_\alpha \leq Q_{\alpha'-2}$ and X_α is elementary abelian.

Now $X_\alpha \nleq L_{\alpha'}$, for otherwise $\eta(L_\alpha, X_\alpha) \leq 2$, which is against (4.10.7). Since $V_{\alpha'-1} \cap Q_{\alpha-1} \leq S_{\alpha-2\alpha-1}$,

$$[J_{\alpha-2\alpha-1}, V_{\alpha'-1} \cap Q_{\alpha-1}] \leq [Z_{\alpha-2}, S_{\alpha-2\alpha-1}; 3],$$

and so $[\![J_{\alpha-2\alpha-1}, V_{\alpha'-1} \cap Q_{\alpha-1}]\!] \leq 2$. Since $H_{\alpha-1} \leq X_\alpha \leq Q_{\alpha'-2} \leq L_{\alpha'-1}$, $V_{\alpha'-1} \cap Q_\alpha$ operates quadratically on $H_{\alpha-1}$ and therefore Theorem 2.15(ii) implies for $t \in J_{\alpha-2\alpha-1}$

$$\left[V_{\alpha'-1} : C_{V_{\alpha\setminus -1}}(t)\right] \leq 2^2 2^4 2 = 2^7.$$

Because $\eta(L_\beta, V_\beta) \geq 2$, Theorem 2.15(v) and Proposition 2.11 force $L_{\alpha'-1}T_{\alpha'-1} \cong \mathrm{Aut}(M_{22})$ and at least one of the noncentral $L_{\alpha'-1}$-chief factors in $V_{\alpha'-1}$ to be a 10-dimensional $\mathrm{Aut}(M_{22})$-module. But then, Proposition 2.11(i) implies that $|(V_{\alpha'-1} \cap Q_\alpha)Q_{\alpha-1}/Q_{\alpha-1}| \leq 2^2$ and so we obtain $[V_{\alpha'-1} : C_{V_{\alpha\setminus -1}}(t)] \leq 2^2 2^2 2 = 2^5$, which gives $\eta(L_\beta, V_\beta) \leq 1$, a contradiction.

From here on we assume that $b = 2$. We also introduce a further normal subgroup of L_α. Suppose that $(\alpha, \alpha') \in \mathcal{C}$. Then set

$$W_\alpha = \left\langle \left[V_\beta, Q_\alpha\right]^{L_\alpha} \right\rangle.$$

LEMMA 4.11. *$\eta(L_\alpha, W_\alpha/Z_\alpha) \geq 1$ and $Q_\beta \cap Q_\alpha$ is not normal in L_α.*

Proof. Suppose first that $W_\alpha = [V_\beta, Q_\alpha]Z_\alpha$. Then $W_\alpha \leq Q_\alpha \cap Q_\beta$ and so $Q_\alpha \cap Q_\beta$ is normal in L_α. So to prove Lemma 4.11 it suffices to show that $X_\alpha := Q_\alpha \cap Q_\beta$ being normal in L_α leads to a contradiction. Clearly, as $b = 2$, we have $X_\alpha \leq L_{\alpha'}$ and $X_\alpha Q_{\alpha'}/Q_{\alpha'} \geq Z_\alpha Q_{\alpha'}/Q_{\alpha'}$. If $L_\alpha/Q_\alpha \cong L_2(q_\alpha)$, then $X_\alpha Q_{\alpha'} = Z_\alpha Q_{\alpha'}$ and thus $[X_\alpha, Z_{\alpha'}] = [Z_\alpha, Z_{\alpha'}] \leq Z_\alpha$ from which we infer that $\eta(L_\alpha, X_\alpha/Z_\alpha) = 0$. Hence $\eta(L_\alpha, Q_\alpha) = 1$. Now suppose that $L_\alpha/Q_\alpha \cong S_5$. If Z_α is the natural S_5-module, then $E(2^2) \cong [Z_\alpha, Z_{\alpha\setminus}] \leq C_{Z_{\alpha\setminus}}(X_\alpha)$. For the case Z_α is the orthogonal S_5-module, Lemmas 4.3(iv) and 2.3(iv) and $Z_\beta \leq Z_\alpha \cap Z_{\alpha'}$ imply that $E(2^2) \cong \langle Z_\beta, [Z_\alpha, Z_{\alpha'}]\rangle \leq C_{Z_{\alpha'}}(X_\alpha) \cap Z_\alpha$. So in either case we deduce that $[X_\alpha, Z_{\alpha'}] \leq Z_\alpha$ and, therefore, $\eta(L_\alpha, Q_\alpha) = 1$ when $L_\alpha/Q_\alpha \cong S_5$. Since $\eta(L_\alpha, Q_\alpha/\Phi(Q_\alpha)) \geq 1$ and $C_{Q_\alpha}(L_\alpha) = 1$, Q_α must be elementary abelian and so Q_α acts quadratically on V_β. If $L_\alpha/Q_\alpha \cong L_2(q_\alpha)$, then $S_{\alpha\beta} = Q_\alpha Q_\beta$, while if $L_\alpha/Q_\alpha \cong S_5$, then $[S_{\alpha\beta} : Q_\alpha Q_\beta] \leq 4$ and in both cases Theorem 2.15(i) provides the desired contradiction.

Set $R_\alpha = \mathrm{core}_{L_\alpha}(V_\beta)$. Because of Lemma 4.11, $V_\beta \cap Q_\alpha > R_\alpha$. Hence we may choose a nontrivial element x in $\Omega_1(Z(S_{\alpha\beta}/R_\alpha)) \cap (V_\beta \cap Q_\alpha)/R_\alpha$ and we then define

$$N_\alpha = \langle x^{L_\alpha} \rangle R_\alpha.$$

Observe that we have $\eta(L_\alpha, N_\alpha/R_\alpha) \neq 0$.

LEMMA 4.12. *N_α/R_α is elementary abelian and $N_\alpha Q_\beta/Q_\beta \le \Omega_1(Z(Q_\alpha Q_\beta/Q_\beta))$.*

Proof. Since $xR_\alpha/R_\alpha \in \Omega_1(Z(Q_\alpha/R_\alpha))$, it follows that N_α/R_α is elementary abelian and that $[N_\alpha, Q_\alpha] \le R_\alpha \le V_\beta \le Q_\beta$. Hence $N_\alpha Q_\beta/Q_\beta \le \Omega_1(Z(Q_\alpha Q_\beta/Q_\beta))$ and so Lemma 4.12 holds.

LEMMA 4.13. *Suppose that $F_\beta/T_\beta \not\cong J_1$ or Fi_{23}. Then $L_\alpha/Q_\alpha \cong L_2(2)$.*

Proof. Put $\hat{N}_\alpha = N_\alpha/R_\alpha$. Suppose first that $S_{\alpha\beta} = Q_\alpha Q_\beta$ holds. We show that it then follows that $L_\alpha/Q_\alpha \cong L_2(2)$. Since $Z_{\alpha'}Q_\alpha \lhd Q_\alpha Q_\beta = S_{\alpha\beta}$, Lemmas 2.3(iv) and 4.3(iv) imply that when $L_\alpha/Q_\alpha \cong S_5$, $Z_{\alpha'}Q_\alpha/Q_\alpha \cong E(2^2)$. Thus (note that $[N_\alpha, Z_\alpha] = 1$ for the $L_\alpha/Q_\alpha \cong S_5$ case) $N_\alpha \cap Q_\beta \le Z_\alpha Q_{\alpha'}$ and so

$$[N_\alpha \cap Q_\beta, Z_{\alpha'}] \le [Z_\alpha, Z_{\alpha'}] \le Z_\alpha.$$

Combining $Q_\alpha Q_\beta = S_{\alpha\beta}$, Lemmas 4.12 and 2.22 gives $[\hat{N}_\alpha : C_{\hat{N}_\alpha}(Z_{\alpha'})] \le 2$. Now $\eta(L_\alpha, \hat{N}_\alpha) \ne 0$ and $Z_{\alpha'}Q_\alpha/Q_\alpha \cong E(2^2)$ (when $L_\alpha/Q_\alpha \cong S_5$) rules out $L_\alpha/Q_\alpha \cong S_5$. Then an appeal to Lemmas 4.3(ii) and 2.31(i) yields $L_\alpha/Q_\alpha \cong L_2(2)$.

Now to complete the proof of the lemma we show that $Q_\alpha Q_\beta \ne S_{\alpha\beta}$ is untenable. So suppose that $Q_\alpha Q_\beta \ne S_{\alpha\beta}$. Then, using Lemma 2.3(iv),

(4.13.1) $L_\alpha/Q_\alpha \cong S_5$ and we may suppose $(\alpha, \alpha') \in \mathbf{C}$ chosen so that $V_\beta Q_\alpha = \langle Z_{\alpha'} Q_\alpha^{S_{\alpha\beta}}\rangle = Q_\alpha Q_\beta$ has index 2 in $S_{\alpha\beta}$ and V_β acts quadratically on Z_α.

(4.13.2) $\eta(L_\beta, [V_\beta, Q_\beta]) \ne 0$.

If (4.13.2) is false, then, by Lemma 3.22 and (4.13.1), $Q_\alpha Q_\beta \in Syl_2(F_\beta)$ and $V_\beta Q_\alpha = Z_{\alpha'}Q_\alpha$. Then we may argue as above to get $[\hat{N}_\alpha : C_{\hat{N}_\alpha}(Z_{\alpha'})] \le 2$. Hence, by (4.13.1), $\eta(L_\alpha, \hat{N}_\alpha) = 0$, a contradiction. So (4.13.2) holds.

(4.13.3) $[V_\beta, Q_\beta] \le K_\alpha$.

First observe from (4.13.1) that $[V_\beta, Q_\beta] \le Q_\alpha$ and $[Z_\alpha, Q_\beta, Q_\beta] = 1$. Hence $[V_\beta, Q_\beta] \le \Omega_1(Z(Q_\beta))$. Now suppose that $[V_\beta, Q_\beta] \nleq K_\alpha$. Then we may select an $\alpha - 1 \in \Delta(\alpha)$ such that $[V_\beta, Q_\beta] \nleq Q_{\alpha-1}$. A standard argument now shows that $[V_\beta, Q_\beta]$ is an FF-module for $L_\beta/C_{L_\beta}([V_\beta, Q_\beta])$, contrary to Theorem 2.15(iii). Therefore, $[V_\beta, Q_\beta] \le K_\alpha$ and we have (4.13.3).

Put $M_\alpha = \langle [V_\beta, Q_\beta]^{L_\alpha}\rangle$. Clearly, by (4.13.3), $M_\alpha \le K_\alpha Q_{\alpha'} \le Q_\beta Q_{\alpha'} \le S_{\beta\alpha'}$ and so $[M_\alpha : C_{M_\alpha}(Z_{\alpha'})] \le 2^2$, from which we see that $\eta(L_\alpha, M_\alpha) \le 2$. In particular, we have $Z_\alpha[V_\beta, Q_\beta, Q_\alpha] \lhd L_\alpha$. Therefore, $[V_\beta, Q_\beta, Q_\alpha, Q_\alpha] \lhd L_\alpha$. Since $Z_\alpha \nleq [V_\beta, Q_\beta]$, we infer that $[V_\beta, Q_\beta, Q_\alpha, Q_\alpha] = 1$. But this, as $[S_{\alpha\beta} : Q_\alpha Q_\beta] = 2$, is against (4.13.2) and Proposition 2.16(i). Thus $Q_\alpha Q_\beta \ne S_{\alpha\beta}$ is untenable, and Lemma 4.13 is proven.

LEMMA 4.14. *If* $L_\beta/Q_\beta \cong J_1$ *or* Fi_{23}, *then* $L_\alpha/Q_\alpha \cong L_2(2)$.

Proof. Aiming for a contradiction we suppose that $L_\beta/Q_\beta \cong J_1$ or Fi_{23} and $q_\alpha \geq 2^2$. Again put $\hat{N}_\alpha = N_\alpha/R_\alpha$. First we examine the case when $L_\alpha/Q_\alpha \cong L_2(q_\alpha)$. Fix $\lambda \in \Delta(\alpha)\setminus\{\beta\}$ so that $\langle Q_\beta, Q_\lambda\rangle Q_\alpha/Q_\alpha = L_\alpha/Q_\alpha$. Then combining Lemma 4.12 with Lemma 2.22 and $[N_\alpha \cap Q_\beta, Z_{\alpha'}] \leq Z_\alpha$ we obtain

$$q_\alpha \leq \left[\hat{N}_\alpha : C_{\hat{N}_\alpha}(Z_{\alpha'})\right] \leq \begin{cases} 2^3 & \text{if } L_\beta/Q_\beta \cong J_1, \\ 2^2 & \text{if } L_\beta/Q_\beta \cong Fi_{23}. \end{cases} \qquad (*)$$

Suppose that $L_\beta/Q_\beta \cong Fi_{23}$. Then, as $q_\alpha \geq 2^2$ (and using Lemma 2.22), $(*)$ implies that $N_\alpha Q_\beta/Q_\beta = \Omega_1(Z(S_{\alpha\beta}/Q_\beta)) \cong E(2^2)$ and $[N_\alpha, V_\lambda]Q_\beta = N_\alpha Q_\beta$. Select $x \in [N_\alpha, V_\lambda]$ so that xQ_β/Q_β is of class $2A$ in L_β/Q_β. Then, on the one hand, we have $[V_\beta \cap Q_\alpha \cap Q_\lambda, V_\lambda] \leq Z_\lambda$ has order at most 2^2 and, by Lemma 2.2, $[V_\beta : V_\beta \cap Q_\alpha \cap Q_\lambda] \leq 2^{2+11}$ giving $[V_\beta : C_{V_\beta}(x)] \leq 2^{15}$, while on the other hand, we have from Lemma 2.17(ii) that $[V_\beta : C_{V_\beta}(x)] \geq 2^{54}$, these two statements being incompatible we deduce that $L_\beta/Q_\beta \cong J_1$.

Since $[V_\beta, K_\alpha] \leq [V_\beta, Q_\beta] = Z_\beta \leq K_\alpha$, $\eta(L_\alpha, K_\alpha) = \eta(L_\alpha, Z_\alpha) = 1$, which then forces K_α to be elementary abelian. Note that $V_\beta \cap Q_\alpha \nleq K_\alpha$ else we obtain $[V_\beta, Q_\alpha] \leq K_\alpha$, against $\eta(L_\alpha, Q_\alpha/K_\alpha) \neq 0$ by Lemma 4.11. So we select $x \in (V_\beta \cap Q_\alpha)\setminus K_\alpha$. Then, as $|[V_\lambda \cap Q_\alpha \cap Q_\beta, x]| \leq |Z_\beta| \leq q_\alpha \leq 2^3$ and $[V_\lambda : V_\lambda \cap Q_\alpha \cap Q_\beta] \leq |S_{\alpha\beta}/Q_\beta| q_\alpha$, we infer that $[V_\lambda : C_{V_\lambda}(x)] \leq q_\alpha^2.2^3$ and so Lemma 2.17(i) implies that $q_\alpha = 2^3$. Therefore, as $[Q_\alpha : Q_\alpha \cap Q_\beta] = 2^3$ and $K_\alpha = Q_\alpha \cap Q_\beta \cap Q_\lambda$, Lemma 4.11 implies that Q_α/K_α is a noncentral L_α/Q_α-chief factor and $\Phi(Q_\alpha) \leq K_\alpha$. Since $[x, Q_\beta] \leq Z_\beta \leq Z_\alpha \leq K_\alpha$, we get $\langle x\rangle K_\alpha \lhd Q_\alpha Q_\beta = S_{\alpha\beta}$ and noting that $Z_\alpha \leq C_{K_\alpha}(x) \leq K_\alpha$, $\eta(L_\alpha, K_\alpha/Z_\alpha) = 0$ then gives $C_{K_\alpha}(x) \lhd L_\alpha$. Because, $K_\alpha\langle x^{L_\alpha}\rangle = Q_\alpha$ we deduce that $Z_\alpha \leq C_{K_\alpha}(x) \leq \Omega_1(Z(Q_\alpha))$. From $q_\alpha = 2^3$, we have $[Q_\beta : C_{Q_\beta}(\Omega_1(Z(Q_\alpha)))] = 2^3$ and so Lemma 2.17 forces $\Omega_1(Z(Q_\alpha)) \leq Q_\beta \leq L_{\alpha'}$. Symmetrically we have $\Omega_1(Z(Q_{\alpha'})) \leq L_\alpha$ and now a standard argument shows that $\Omega_1(Z(Q_\alpha))$ is an FF-module for L_α/Q_α, thence, by Lemma 2.32, $\Omega_1(Z(Q_\alpha)) = Z_\alpha$. Therefore, $C_{K_\alpha}(x) = \Omega_1(Z(Q_\alpha)) = Z_\alpha$. Because $x^2 \in K_\alpha$, x acts as an involution on the elementary abelian group K_α. Thus

$$q_\alpha = |Z_\beta| \geq |[x, K_\alpha]| = \left[K_\alpha : C_{K_\alpha}(x)\right] = [K_\alpha : Z_\alpha].$$

Consequently, as $q_\alpha = 2^3$,

$$|S_{\alpha\beta}| = \left[S_{\alpha\beta} : Q_\alpha\right][Q_\alpha : K_\alpha][K_\alpha : Z_\alpha]|Z_\alpha| \leq q_\alpha q_\alpha^2 q_\alpha q_\alpha^2 = q_\alpha^6 = 2^{18}.$$

Finally, as $L_\alpha/Q_\beta \cong J_1$, we get $|Q_\beta| \leq 2^{15}$ and hence, courtesy of Lemma 2.17, $\eta(L_\beta, Q_\beta) = 0$, a contradiction.

Now we consider the remaining case $L_\alpha/Q_\alpha \cong S_5$. Since $L_\beta = F_\beta$, applying Lemma 3.22 gives $\eta(L_\beta, V_\beta) \geq 2$. Choose n maximal so that $\eta(L_\beta, [V_\beta, Q_\beta; n]) \neq 0$. Set $R_\beta = [V_\beta, Q_\beta; n]$. From the maximal choice of n, $[R_\beta, Q_\beta] \leq Z_\beta$ and so $\|[R_\beta, Q_\beta]\| \leq 2^3$.

Suppose that $R_\beta \cap Q_\alpha \nleq K_\alpha$. Then we can find $\alpha - 1 \in \Delta(\alpha)$ such that $R_\beta \cap Q_\alpha \nleq Q_{\alpha-1}$. Let $y \in (R_\beta \cap Q_\alpha)\backslash Q_{\alpha-1}$. Now we have

$$[V_{\alpha-1} : V_{\alpha-1} \cap Q_\alpha \cap Q_\beta] \leq \begin{cases} 2^{3+3} & \text{if } L_\beta/Q_\beta \cong J_1, \\ 2^{3+18} & \text{if } L_\beta/Q_\beta \cong Fi_{23}. \end{cases} \qquad (*)$$

Since $\|[R_\beta, Q_\beta]\| \leq 2^3$ and $\eta(L_\beta, V_\beta) \geq 2$, by $(*)$ there exists a non-central $L_{\alpha-1}$-chief factor, V, within $V_{\alpha-1}$ for which

$$[V : C_V(y)] \leq \begin{cases} 2^4 & \text{if } L_\beta/Q_\beta \cong J_1, \\ 2^{12} & \text{if } L_\beta/Q_\beta \cong Fi_{23}. \end{cases}$$

Both these possibilities are ruled out by Lemma 2.17. We therefore deduce that $R_\beta \cap Q_\alpha \leq K_\alpha$. If $R_\beta \nleq Q_\alpha$, then it follows that $\eta(L_\alpha, Q_\alpha/K_\alpha) = 0$, against Lemma 4.11. So $R_\beta \leq Q_\alpha$ and thus $R_\beta \leq K_\alpha$. Also, observe that $R_\beta \leq Z(V_\beta)$. Now select $\mu \in \Delta(\alpha)$ so that $L_\alpha = \langle S_{\mu\alpha}, Z_{\alpha'} \rangle$. Since $R_\mu \leq K_\alpha$, $R_\mu \leq Q_\beta \leq L_{\alpha'}$. If $Z_{\alpha'}$ is the natural S_5-module, then $R_\mu \leq Z_\alpha Q_{\alpha'}$. While if Z_α is the orthogonal S_5-module, then either $R_\mu Q_{\alpha'}/Q_{\alpha'} = Z_\alpha Q_{\alpha'}/Q_{\alpha'} \cong E(2)$ or $C_{Z_\alpha \backslash}(R_\mu) = \langle Z_\beta, [Z_\alpha, Z_{\alpha'}] \rangle$. So in any case $[R_\mu, Z_{\alpha'}] \leq Z_\alpha$. Therefore, $R_\mu Z_\alpha \lhd L_\alpha$. Commutating with Q_α reveals, as $Z_\alpha \nleq R_\mu$, that $R_\mu \leq \Omega_1(Z(Q_\alpha))$. But then $\eta(L_\mu, R_\mu) = 0$, a contradiction. This contradiction concludes the proof that if $L_\beta/Q_\beta \cong J_1$ or Fi_{23}, then $q_\alpha = 2$.

Using the language of [PR2], Lemmas 4.10, 4.11, 4.13, and 4.14 show that any group G, satisfying Hypothesis 4.1, is an *amalgam of symplectic type over* $GF(2)$. Consequently the assertions in Theorem 4.2 can be read from [PR2, Main Theorem], and Theorem C is proven.

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